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Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights

Bill Jackson
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, England
B.JACKSON@QMUL.AC.UK

Aldo Procacci
Departamento de Matemática
Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627 – Caixa Postal 702
30161-970 Belo Horizonte, MG – BRASIL
ALDO@MAT.UFMG.BR

Alan D. Sokal*
Department of Physics
New York University
4 Washington Place
New York, NY 10003 USA
SOKAL@NYU.EDU

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Abstract

We find zero-free regions in the complex plane at large $|q|$ for the multivariate Tutte polynomial (also known in statistical mechanics as the Potts-model partition function) $Z_G(q, \boldsymbol{w})$ of a graph G with general complex edge weights $\boldsymbol{w} = \{w_e\}$. This generalizes a result of Sokal [27] that applied only within the complex antiferromagnetic regime $|1+w_e| \le 1$. Our proof uses the polymer-gas representation of the multivariate Tutte polynomial together with the Penrose identity.

Key Words: Graph, chromatic polynomial, multivariate Tutte polynomial, Potts model, Penrose identity, Penrose inequality, Lambert W function.

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*Also at Department of Mathematics, University College London, London WC1E 6BT, England.

1 Introduction

A decade ago, Sokal [27] proved that if $G = (V, E)$ is a loopless graph[1] of maximum degree Δ, then all the roots (real or complex) of the chromatic polynomial $P_G(q)$ lie in the disc $|q| < C(\Delta)$, where $C(\Delta)$ are semi-explicit constants (given by a variational formula) satisfying $C(\Delta) \leq 7.963907\Delta$.[2] More generally, Sokal proved a bound on the zeros of the multivariate Tutte polynomial [29] (also known in statistical mechanics as the Potts-model partition function [25, 33, 34])

$$Z_G(q, \boldsymbol{w}) \;=\; \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e \tag{1.1}$$

[here $k(A)$ denotes the number of connected components in the subgraph (V, A)] when the edge weights $\boldsymbol{w} = \{w_e\}$ lie in the "complex antiferromagnetic regime" $|1+w_e| \leq 1$:

Theorem 1.1 [27, Corollary 5.5] *Let $G = (V, E)$ be a loopless graph equipped with complex edge weights $\boldsymbol{w} = \{w_e\}_{e \in E}$ satisfying $|1 + w_e| \leq 1$ for all e. Then all the zeros of $Z_G(q, \boldsymbol{w})$ lie in the disc $|q| < K\Delta(G, \boldsymbol{w})$, where*

$$\Delta(G, \boldsymbol{w}) \;=\; \max_{x \in V} \sum_{e \ni x} |w_e| \tag{1.2}$$

and

$$\begin{aligned} K \;&=\; \min \left\{ L \colon\; \inf_{\alpha > 0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n} \, L^{-(n-1)} \, \frac{n^{n-1}}{n!} \;\leq\; 1 \right\} && \text{(1.3a)} \\ &=\; \min_{a > 0} \frac{a + e^a}{\log(1 + a e^{-a})} && \text{(1.3b)} \\ &\approx\; 7.963\,906\,075\,890\,002\,502\;\ldots\;. && \text{(1.3c)} \end{aligned}$$

Moreover, we rigorously have $K \leq 7.963907$.

Here the simpler formula (1.3b) for the constant K is due to Borgs [9, Theorem 2.1].

The purpose of this paper is to extend Sokal's bound by removing the condition that $|1 + w_e| \leq 1$ for all e. More precisely, we shall prove:

[1] All graphs in this paper are finite and undirected; furthermore, they are *allowed* to contain loops and multiple edges unless we explicitly state otherwise.

[2] More recently, Borgs [9] has provided a simpler variational characterization of the constant $K = \lim_{\Delta \to \infty} C(\Delta)/\Delta \approx 7.963906$ than the one given by Sokal [27, Proposition 5.4] — compare eqs. (1.3a) and (1.3b) below — and Fernández and Procacci [14] have provided, in an analogous way, a simpler variational characterization of the constants $C(\Delta)$. Furthermore, Fernández and Procacci [14] have improved the constants $C(\Delta)$ to smaller constants $C^*(\Delta)$, for which $K^* = \lim_{\Delta \to \infty} C^*(\Delta)/\Delta \approx 6.907652$.

Theorem 1.2 *Let $G = (V, E)$ be a loopless graph equipped with complex edge weights $\boldsymbol{w} = \{w_e\}_{e \in E}$. Then all the zeros of $Z_G(q, \boldsymbol{w})$ lie in the disc*

$$|q| \;<\; \mathcal{K}^*(\Psi(G, \boldsymbol{w}))\, \Delta'(G, \boldsymbol{w}) \;, \tag{1.4}$$

where

$$\Delta'(G, \boldsymbol{w}) \;=\; \max_{x \in V} \sum_{e \ni x} \min\left\{ |w_e|, \, \frac{|w_e|}{|1 + w_e|} \right\} \tag{1.5}$$

$$\Psi(G, \boldsymbol{w}) \;=\; \max_{x \in V} \prod_{e \ni x} \max\{1, |1 + w_e|\} \tag{1.6}$$

and

$$\mathcal{K}^*(\psi) \;=\; \min\left\{ L\colon \; \inf_{\alpha > 0} \, (e^\alpha - 1)^{-1} \sum_{n=2}^{\infty} e^{\alpha n} \, \psi^{n/2} \, L^{-(n-1)} \, \frac{n^{n-1}}{n!} \;\le\; 1 \right\} \tag{1.7a}$$

$$= \min_{1 < y < 1 + \psi^{-1/2}} \frac{y}{(1 + \psi^{-1/2} - y) \log y} \tag{1.7b}$$

$$= W\!\left(\frac{e}{1 + \psi^{-1/2}} \right) \Big/ \left[1 - W\!\left(\frac{e}{1 + \psi^{-1/2}} \right) \right]^2 \tag{1.7c}$$

$$\le\; 4\psi + 3\psi^{1/2} \;, \tag{1.7d}$$

where W is the Lambert W function [11], i.e. the inverse function to $x \mapsto x e^x$.

When $|1 + w_e| \le 1$ for all e, we have $\Delta'(G, \boldsymbol{w}) = \Delta(G, \boldsymbol{w})$ and $\Psi(G, \boldsymbol{w}) = 1$, so that Theorem 1.2 reduces in this case to Theorem 1.1 with an improved constant [14] $K^* \equiv \mathcal{K}^*(1) = W(e/2)/[1 - W(e/2)]^2 \approx 6.907\,651\,697\,774\,449\,218\,\ldots$. This explicit formula for the Fernández–Procacci [14] constant K^* appears to be new.

Let us also remark that the upper bound (1.7d) gives precisely the first two terms of the large-ψ asymptotics of $\mathcal{K}^*(\psi)$: see (6.36).

Please note that $\Psi(G, \boldsymbol{w})$ involves a *product* over edges $e \ni x$ rather than a sum, and hence grows *exponentially* (rather than linearly) with the vertex degree whenever $|1 + w_e| > 1$. The resulting exponential dependence of the bound on $|q|$ given in Theorem 1.2 is not merely an artifact of our proof, but is a genuine feature of the regime $|1 + w_e| > 1$.[3] To see this, it suffices to note that whenever one replaces an edge e by k edges in parallel, the effective couplings $w_{e,\mathrm{eff}} = (1 + w_e)^k - 1$ grow exponentially in k when $|1 + w_e| > 1$ but only linearly when $|1 + w_e| \le 1$. For instance, the graph $G = K_2^{(k)}$ (a pair of vertices connected by k parallel edges) with all edge weights equal has $Z_G(q, w) = q[q + (1 + w)^k - 1]$, so that we must take $|q| > |(1 + w)^k - 1|$ to avoid a root. This has roughly (but not exactly) the same

[3] See also [27, Remark 2 after Corollary 5.5].

dependence in w and k as the bound of Theorem 1.2. See Example 7.3 below for details.

The *linear* growth of the bound (1.4)/(1.7) as $\Psi(G, \boldsymbol{w}) \to \infty$ is, however, an artifact of allowing multiple edges. If we restrict attention to *simple* graphs, then with a little more combinatorial work we can obtain a bound that grows only like $\Psi(G, \boldsymbol{w})^{1/2}$:

Theorem 1.3 *Let $G = (V, E)$ be a simple graph (i.e. no loops or multiple edges) equipped with complex edge weights $\boldsymbol{w} = \{w_e\}_{e \in E}$. Then all the zeros of $Z_G(q, \boldsymbol{w})$ lie in the disc*

$$|q| \;<\; K_\lambda^* \, \Psi(G, \boldsymbol{w})^{1/2} \, \widetilde{\Delta}(G, \boldsymbol{w}) \,, \tag{1.8}$$

where

$$\widetilde{\Delta}(G, \boldsymbol{w}) = \max_{x \in V} \sum_{e \ni x} \min \left\{ |w_e|, \, \frac{|w_e|}{|1 + w_e|^{1/2}} \right\} \tag{1.9}$$

and $\lambda = \Delta'(G, \boldsymbol{w}) / \widetilde{\Delta}(G, \boldsymbol{w})$ and

$$K_\lambda^* \;=\; \min \left\{ L\colon \; \inf_{\alpha > 0} (e^\alpha - 1)^{-1} \sum_{n=2}^{\infty} e^{\alpha n} L^{-(n-1)} \, \frac{[1 + (n-1)\lambda]^{n-2}}{(n-1)!} \;\le\; 1 \right\} \tag{1.10a}$$

$$\;=\; \min_{1 < y < 2} \frac{y^\lambda}{(2 - y) \log y} \tag{1.10b}$$

$$\;\le\; 5 + 2\lambda \,. \tag{1.10c}$$

The constant K_λ^* is an increasing function of $\lambda = \Delta'(G, \boldsymbol{w}) / \widetilde{\Delta}(G, \boldsymbol{w}) \in (0, 1]$, but the variation is fairly weak. In the complex antiferromagnetic regime $|1 + w_e| \le 1$, where $\lambda = 1$, we have $K_1^* = K^* \approx 6.907652$, i.e. the same constant as in Theorem 1.2 (indeed, the two theorems give the same bound in this case); while in the complex ferromagnetic regime $|1 + w_e| > 1$, where $0 < \lambda < 1$, K_λ^* decreases as $|w_e| \to \infty$ down to the value $K_0^* = W(2e)/[2\,[W(2e) - 1]^2] \approx 4.892888$. It is worth noting that the weaker version of Theorem 1.3 in which K_λ^* and $\widetilde{\Delta}(G, \boldsymbol{w})$ are replaced by K_1^* and $\Delta(G, \boldsymbol{w})$, respectively, can be proven by a slightly simpler argument (see the Remark in Section 6 after the proof of Theorem 1.3).[4]

Please observe that in the bound (1.8) we pay a price, compared to (1.4), by having $\widetilde{\Delta}(G, \boldsymbol{w})$ in place of $\Delta'(G, \boldsymbol{w})$. In fact, the simple example $G = K_2$ shows that the bound of Theorem 1.3 can in some cases be inferior to that of Theorem 1.2, by a factor of up to $K_0^*/4 \approx 1.223222$ (see Examples 7.1 and 7.2 below). But this seems to be the largest possible ratio of the two bounds; and in any event the ratio cannot exceed $K_1^*/4 \approx 1.726913$ (see the Discussion in Section 6 after the proof of Theorem 1.3). In most cases, however, Theorem 1.3 is a big improvement over Theorem 1.2, due

[4] The first preprint version of this paper (`http://arxiv.org/abs/0810.4703v1`) contained only this weakened version of Theorem 1.3.

to its replacement of $\Psi(G,\boldsymbol{w})$ by $\Psi(G,\boldsymbol{w})^{1/2}$: see, for instance, Examples 7.5 and 7.6. Furthermore, the growth as $\Psi(G,\boldsymbol{w})^{1/2}$ is essentially best possible: see again Examples 7.5 and 7.6.

It is curious that the bound of Theorem 1.3 is not always better than that of Theorem 1.2, despite using better "ingredients" in its proof; the reasons for this will be discussed near the end of Section 6. It would be interesting to try to find a single natural bound that simultaneously improves Theorems 1.2 and 1.3.

Please note also (see e.g. [29]) that if G is a loopless graph with multiple edges, then its multivariate Tutte polynomial is identical to that of the underlying simple graph $\widehat{G}$ in which each set of parallel edges $e_1,\ldots,e_k$ in G is replaced by a single edge e in $\widehat{G}$ with weight $\widehat{w}_e = \prod_{i=1}^k (1+w_{e_i}) - 1$. So one is always free to apply Theorems 1.2 and 1.3 to $(\widehat{G},\widehat{\boldsymbol{w}})$ instead of applying Theorem 1.2 to $(G,\boldsymbol{w})$. The following lemma concerning the behavior of $\Psi(G,\boldsymbol{w})$ and $\Delta'(G,\boldsymbol{w})$ under parallel reduction — which will be proven at the end of Section 6 — implies that the bound we get by applying Theorem 1.2 to $(\widehat{G},\widehat{\boldsymbol{w}})$ will never be worse than the bound we get by applying Theorem 1.2 to $(G,\boldsymbol{w})$. So we can find our best bound for any given (multi)graph G by constructing $(\widehat{G},\widehat{\boldsymbol{w}})$ and then taking the minimum of the bounds we obtain by applying (1.4) and (1.8) to $(\widehat{G},\widehat{\boldsymbol{w}})$.

Lemma 1.4 *Let $w_1, w_2 \in \mathbb{C}$ and put $w_3 = (1+w_1)(1+w_2) - 1$. Then*

$$\max\{1, |1+w_3|\} \;\le\; \max\{1, |1+w_1|\}\, \max\{1, |1+w_2|\} \tag{1.11}$$

and

$$\min\left\{ |w_3|,\, \frac{|w_3|}{|1+w_3|} \right\} \;\le\; \min\left\{ |w_1|,\, \frac{|w_1|}{|1+w_1|} \right\} \;+\; \min\left\{ |w_2|,\, \frac{|w_2|}{|1+w_2|} \right\} . \tag{1.12}$$

Sokal's proof of Theorem 1.1 involved the following steps:

1. Write the multivariate Tutte polynomial $Z_G(q,\boldsymbol{w})$ as the partition function of a polymer gas with weights depending on q and $\boldsymbol{w}$ (this is easy: see Section 2 below).

2. Invoke the Kotecký–Preiss [20] condition for the nonvanishing of the partition function of a polymer gas.

3. Control the polymer weights by bounding sums over connected subgraphs by sums over trees, using the Penrose inequality [24]. This step required $|1+w_e| \le 1$.

4. Bound the total weight of n-vertex trees (or more generally, of connected subgraphs with m edges) in G that contain a specified vertex $x \in V$.

5. Put everything together to prove that $Z_G(q,\boldsymbol{w}) \ne 0$ whenever q lies outside a specified disc.

Here we follow the same outline, but modify step 3 so as to allow arbitrary complex weights w_e. In addition, in step 2 we replace the Kotecký–Preiss condition by the more powerful Gruber–Kunz–Fernández–Procacci [16, 13] condition, thereby slightly improving the numerical constant along the lines of the work of Fernández and Procacci [14] for chromatic polynomials. Finally, Theorem 1.3 needs a slightly strengthened version of the bound in step 4.

The plan of this paper is to treat each of these five steps in successive sections. Thus, in Section 2 we recall how the multivariate Tutte polynomial $Z_G(q, \boldsymbol{w})$ can be written as the partition function of a polymer gas. In Section 3 we recall the Kotecký–Preiss and Gruber–Kunz–Fernández–Procacci conditions for the nonvanishing of the partition function of a polymer gas. In Section 4 we recall the Penrose identity [24] and show how to use it to bound the polymer weights *without* assuming that $|1+w_e| \le 1$; this is our main new contribution. In Section 5 we recall the sharp bound [27, 17] in terms of maximum weighted degree $\Delta(G, \boldsymbol{w})$ on the total weight of connected m-edge subgraphs in G that contain a specified vertex x; we also strengthen it slightly by taking specific account of the edges incident on x. In Section 6 we put everything together to prove Theorems 1.2 and 1.3; we also prove Lemma 1.4. Finally, in Section 7 we examine some examples that shed light on the extent to which Theorems 1.2 and 1.3 are sharp or non-sharp.

2 Polymer-gas representation of $Z_G(q, w)$

In this section we recall how to rewrite the multivariate Tutte polynomial $Z_G(q, \boldsymbol{w})$ as the partition function of a polymer gas living on the vertex set of G. This easy result is due to Sokal and Kupiainen [27, Proposition 2.1].

First, some notation: If $H = (\mathsf{V}, \mathsf{E})$ is a graph equipped with edge weights $\boldsymbol{w} = \{w_e\}_{e \in \mathsf{E}}$, we denote by $C_H(\boldsymbol{w})$ the generating polynomial of connected spanning subgraphs of H, i.e.

$$C_H(\boldsymbol{w}) \;=\; \sum_{\substack{A \subseteq \mathsf{E} \\ (\mathsf{V}, A) \text{ connected}}} \prod_{e \in A} w_e \,. \tag{2.1}$$

Note that $C_H(\boldsymbol{w}) \equiv 0$ if H is disconnected.

If $G = (V, E)$ is a graph and $S \subseteq V$, we denote by $G[S]$ the induced subgraph of G on S, i.e. $G[S]$ is the graph whose vertex set is S and whose edges consist of all the edges of G both of whose endpoints lie in S.

Proposition 2.1 (polymer representation of the multivariate Tutte polynomial) *Let $G = (V, E)$ be a loopless graph equipped with edge weights $\boldsymbol{w} = \{w_e\}_{e \in E}$. Then*

$$q^{-|V|} Z_G(q, \boldsymbol{w}) \;=\; \sum_{N=0}^{\infty} \sum_{\substack{\{S_1, \ldots, S_N\} \\ \text{disjoint}}} \prod_{i=1}^{n} \xi(S_i) \,, \tag{2.2}$$

where the sum runs over unordered collections $\{S_1, \ldots, S_N\}$ *of disjoint nonempty subsets of* V*, and the weights* $\xi(S)$ *are given by*

$$\xi(S) \;=\; \begin{cases} q^{-(|S|-1)} C_{G[S]}(\boldsymbol{w}) & \textit{if } |S| \geq 2 \\ 0 & \textit{if } |S| = 1 \end{cases} \tag{2.3}$$

[The $N = 0$ *term in the sum (2.2) is understood to contribute* 1*.]*

The identity (2.2) thus represents $q^{-|V|} Z_G(q, \boldsymbol{w})$ as the partition function of a gas of nonoverlapping "polymers" living on V, with weights (2.3). Here a "polymer" is, in principle, any nonempty subset $S \subseteq V$; but since the weight $\xi(S)$ vanishes for sets of cardinality 1, we can equivalently restrict our polymers to be subsets of cardinality at least 2. Likewise, the weight $\xi(S)$ vanishes whenever the induced subgraph $G[S]$ is disconnected; so we can, if we wish, restrict our polymers to be sets S for which $G[S]$ is connected.

Proof of Proposition 2.1. Starting from the definition (1.1) of $Z_G(q, \boldsymbol{w})$, let us separate the terms in the sum according to the number k of connected components [i.e. $k(A) = k$] and according to the partition $\{S_1, \ldots, S_k\}$ of V that is induced by the vertex sets of those connected components; we will then sum over all ways of choosing edges within those vertex sets S_i so as to connect those vertices. We thus have

$$Z_G(q, \boldsymbol{w}) \;=\; q^{|V|} \sum_{k \geq 1} \sum_{\substack{\{S_1, \ldots, S_k\} \\ V = \biguplus S_i}} \prod_{i=1}^{k} q^{-(|S_i|-1)} C_{G[S_i]}(\boldsymbol{w}) \;, \tag{2.4}$$

where the sum runs over all unordered partitions $\{S_1, \ldots, S_k\}$ of V into nonempty subsets, and we have used $|V| = \sum_{i=1}^{k} |S_i|$. Note now that any set S_i of cardinality 1 gets weight $q^{-(|S_i|-1)} C_{G[S_i]}(\boldsymbol{w}) = 1$ (here we have used the fact that G is loopless). So let us define $\{S'_1, \ldots, S'_N\}$ to be the subcollection of $\{S_1, \ldots, S_k\}$ consisting of the sets of cardinality ≥ 2; and let us note that there is a one-to-one correspondence between unordered partitions $\{S_1, \ldots, S_k\}$ of V into nonempty subsets and unordered collections $\{S'_1, \ldots, S'_N\}$ of disjoint subsets of V of cardinality at least 2 (which need not cover all of V: indeed, the points not covered correspond to the singleton sets S_i in the original partition). Passing to $\{S'_1, \ldots, S'_N\}$ and dropping the primes, we have (2.2)/(2.3). $\square$

3 Sufficient condition for the nonvanishing of a polymer-gas partition function

Let V be a finite set, and let $\{\rho(S)\}_{\varnothing \neq S \subseteq V}$ be a collection of complex weights associated to the nonempty subsets of V. Consider now a gas of nonoverlapping

"polymers" living on V, with weights $\rho(S)$: the partition function of such a polymer gas is, by definition,

$$\Xi \;=\; \sum_{N=0}^{\infty} \sum_{\substack{\{S_1,\ldots,S_N\} \\ \text{disjoint}}} \prod_{i=1}^{n} \rho(S_i) \;, \tag{3.1}$$

where the sum runs over unordered collections $\{S_1, \ldots, S_N\}$ of disjoint nonempty subsets of V, and the $N = 0$ term in (3.1) is understood to contribute 1. The following proposition — essentially proven almost four decades ago by Gruber and Kunz [16, Section 4, cf. eq. (33)] but largely forgotten, and then rediscovered very recently by Fernández and Procacci [13, eq. (3.17)] with a new proof — gives a sufficient condition for the nonvanishing of a polymer-gas partition function:

Proposition 3.1 (Gruber–Kunz–Fernández–Procacci condition) *Let V be a finite set, and let $\{\rho(S)\}_{\varnothing \neq S \subseteq V}$ be complex weights associated to the nonempty subsets of V. Suppose that there exists a number $\alpha > 0$ such that*

$$\sup_{x \in V} \sum_{S \ni x} e^{\alpha |S|} \, |\rho(S)| \;\le\; e^{\alpha} - 1 \,. \tag{3.2}$$

Then

$$\Xi \;\equiv\; \sum_{N=0}^{\infty} \sum_{\substack{\{S_1,\ldots,S_N\} \\ \text{disjoint}}} \prod_{i=1}^{n} \rho(S_i) \;\neq\; 0 \,. \tag{3.3}$$

See also [6] for an extremely simple proof of Proposition 3.1 by induction on V.

In the slightly less powerful Kotecký–Preiss [20] condition, the term $e^{\alpha} - 1$ on the right-hand side of (3.2) is replaced by α.

Remark. Suppose that (as happens in all nontrivial cases) there exists a set S with $|S| \ge 2$ and $\rho(S) \neq 0$. Then the hypothesis that there exists $\alpha > 0$ such that (3.2) holds can be rewritten as

$$\inf_{\alpha > 0} (e^{\alpha} - 1)^{-1} \sup_{x \in V} \sum_{S \ni x} e^{\alpha |S|} \, |\rho(S)| \;\le\; 1 \,, \tag{3.4}$$

since in this case the infimum on the left-hand side of (3.4) will always be attained at some $\alpha > 0$.[5] We will use the Gruber–Kunz–Fernández–Procacci condition in the form (3.4).

[5]If there exists a set S with $|S| \ge 2$ and $\rho(S) \neq 0$, then the function $f(\alpha)$ being minimized on the left-hand side of (3.4) is a continuous function that tends to $+\infty$ as $\alpha \downarrow 0$ and as $\alpha \uparrow \infty$, hence its minimum is attained.

There is one exceptional case in which (3.4) holds but there does not exist $\alpha > 0$ such that (3.2) holds: namely, if $\rho(S) = 0$ whenever $|S| \ge 2$ and in addition we have $\max_{x \in V} |\rho(\{x\})| = 1$. Indeed, if $\rho(S) = 0$ for $|S| \ge 2$, we have $\Xi = \prod_{x \in V} [1 + \rho(\{x\})]$, which vanishes when at least one $\rho(\{x\})$ equals -1; so (3.4) *fails* (barely) to imply $\Xi \neq 0$ in this case.

4 A bound on $C_H(\boldsymbol{w})$ via the Penrose identity

In this section we recall the Penrose identity [24] and show how it can be used to bound a sum over connected subgraphs by a sum over trees *even in the absence of the hypothesis* $|1 + w_e| \le 1$.

Let $H = (\mathsf{V}, \mathsf{E})$ be a graph. Recall that $C_H(\boldsymbol{w})$ denotes the generating polynomial of connected spanning subgraphs of H:

$$C_H(\boldsymbol{w}) \;=\; \sum_{\substack{A \subseteq \mathsf{E} \\ (\mathsf{V},A) \text{ connected}}} \prod_{e \in A} w_e \;. \tag{4.1}$$

We denote by $T_H(\boldsymbol{w})$ the generating polynomial of spanning trees in H:

$$T_H(\boldsymbol{w}) \;=\; \sum_{\substack{A \subseteq \mathsf{E} \\ (\mathsf{V},A) \text{ tree}}} \prod_{e \in A} w_e \;. \tag{4.2}$$

Let $\mathcal{C}$ (resp. $\mathcal{T}$) be the set of subsets $A \subseteq \mathsf{E}$ such that (V, A) is connected (resp. is a tree). Clearly $\mathcal{C}$ is an increasing family of subsets of E with respect to set-theoretic inclusion, and the minimal elements of $\mathcal{C}$ are precisely those of $\mathcal{T}$ (i.e. the spanning trees). It is a nontrivial combinatorial fact — apparently first discovered by Penrose [24] — that the (anti-)complex $\mathcal{C}$ is *partitionable*: that is, there exists a map $\mathbf{R}\colon \mathcal{T} \to \mathcal{C}$ such that $\mathbf{R}(T) \supseteq T$ for all $T \in \mathcal{T}$ and $\mathcal{C} = \biguplus_{T \in \mathcal{T}} [T, \mathbf{R}(T)]$ (disjoint union), where $[E_1, E_2]$ denotes the Boolean interval $\{A\colon\; E_1 \subseteq A \subseteq E_2\}$. In fact, many alternative choices of $\mathbf{R}$ are available[6], and most of our arguments will not depend on any specific choice of $\mathbf{R}$. An immediate consequence of the existence of $\mathbf{R}$ is the following simple but fundamental identity:

Proposition 4.1 (Penrose identity [24]) *Let* $\mathbf{R}\colon \mathcal{T} \to \mathcal{C}$ *be any map such that* $\mathbf{R}(T) \supseteq T$ *for all* $T \in \mathcal{T}$ *and* $\mathcal{C}$ *is the disjoint union of the Boolean intervals* $[T, \mathbf{R}(T)]$ *over* $T \in \mathcal{T}$*. Then*

$$C_H(\boldsymbol{w}) \;=\; \sum_{\substack{T \subseteq \mathsf{E} \\ (\mathsf{V},T) \text{ tree}}} \prod_{e \in T} w_e \sum_{T \subseteq A \subseteq \mathbf{R}(T)} \prod_{e \in A \setminus T} w_e \tag{4.3a}$$

$$\;=\; \sum_{\substack{T \subseteq \mathsf{E} \\ (\mathsf{V},T) \text{ tree}}} \prod_{e \in T} w_e \prod_{e \in \mathbf{R}(T) \setminus T} (1 + w_e) \;. \tag{4.3b}$$

If $|1 + w_e| \le 1$ for all e, then it is obvious that we can take absolute values everywhere in (4.3b) and drop the factors $|1 + w_e|$, yielding:

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[6] See e.g. [24], [7, Sections 7.2 and 7.3], [36, Section 8.3], [15, Sections 2 and 6], [5, Proposition 13.7 et seq.], [27, Proposition 4.1] and [26, Lemma 2.2].

Proposition 4.2 (Penrose inequality [24]) *Let $H = (\mathsf{V}, \mathsf{E})$ be a graph equipped with complex edge weights $\boldsymbol{w} = \{w_e\}_{e\in\mathsf{E}}$ satisfying $|1 + w_e| \le 1$ for all e. Then*

$$|C_H(\boldsymbol{w})| \;\le\; T_H(|\boldsymbol{w}|) \;. \tag{4.4}$$

Remark. By using a specific choice of the map $\mathbf{R}$ (namely, that of Penrose [24]), Fernández and Procacci [13] have recently shown how to improve Proposition 4.2 when $w_e \in \{-1, 0\}$ for all e; and this improvement plays a key role in their proof of the Gruber–Kunz–Fernández–Procacci condition (Proposition 3.1) for polymer gases with hard-core repulsive interactions. See also Fernández *et al.* [12] for a generalization to $-1 \le w_e \le 0$, which leads to an improved convergence criterion for the Mayer expansion in lattice gases with soft repulsive interactions. $\square$

Let us now show what can be done *without* the hypothesis $|1 + w_e| \le 1$.

Proposition 4.3 (extended Penrose inequality) *Let $H = (\mathsf{V}, \mathsf{E})$ be a graph equipped with complex edge weights $\boldsymbol{w} = \{w_e\}_{e\in\mathsf{E}}$. Then*

$$\begin{aligned} |C_H(\boldsymbol{w})| \;&\le\; T_H(\boldsymbol{w}') \prod_{e\in\mathsf{E}} \max\{1, |1 + w_e|\} && (4.5a)\\ &\le\; T_H(\boldsymbol{w}')\, \Psi(H, \boldsymbol{w})^{|\mathsf{V}|/2} \;, && (4.5b) \end{aligned}$$

where

$$w'_e \;=\; \min\left\{ |w_e|, \frac{|w_e|}{|1 + w_e|} \right\} \tag{4.6}$$

and

$$\Psi(H, \boldsymbol{w}) \;=\; \max_{x\in\mathsf{V}} \prod_{e\ni x} \max\{1, |1 + w_e|\} \;. \tag{4.7}$$

Proof. Let us write $\mathsf{E}_+ = \{e \in \mathsf{E} \colon |1 + w_e| > 1\}$. Taking absolute values in the Penrose identity (4.3b), we obviously have

$$|C_H(\boldsymbol{w})| \;\le \sum_{\substack{T\subseteq\mathsf{E} \\ (\mathsf{V},T)\,\text{tree}}} \prod_{e\in T} |w_e| \prod_{e\in\mathbf{R}(T)\setminus T} |1 + w_e| \;. \tag{4.8}$$

Now, in each summand on the right-hand side of (4.8), multiply and divide by $|1+w_e|$

for each $e \in T \cap \mathsf{E}_+$; we have

$$
\begin{aligned}
|C_H(\boldsymbol{w})| \;&\le\; \sum_{\substack{T \subseteq \mathsf{E} \\ (\mathsf{V},T) \text{ tree}}} \prod_{e \in T} |w_e| \prod_{e \in T \cap \mathsf{E}_+} \frac{1}{|1+w_e|} \prod_{e \in T \cap \mathsf{E}_+} |1+w_e| \prod_{e \in \mathbf{R}(T) \setminus T} |1+w_e| && \text{(4.9a)} \\
&=\; \sum_{\substack{T \subseteq \mathsf{E} \\ (\mathsf{V},T) \text{ tree}}} \prod_{e \in T} \min\left\{ |w_e|,\, \frac{|w_e|}{|1+w_e|} \right\} \prod_{e \in (T \cap \mathsf{E}_+) \cup (\mathbf{R}(T) \setminus T)} |1+w_e| && \text{(4.9b)} \\
&\le\; \sum_{\substack{T \subseteq \mathsf{E} \\ (\mathsf{V},T) \text{ tree}}} \prod_{e \in T} \min\left\{ |w_e|,\, \frac{|w_e|}{|1+w_e|} \right\} \prod_{e \in \mathsf{E}} \max\{1, |1+w_e|\} \;. && \text{(4.9c)}
\end{aligned}
$$

This proves (4.5a). Then (4.5b) is a trivial consequence. $\square$

If we assume that the graph H is *simple* (i.e. has no loops or multiple edges), then we can get a slightly better bound:

Proposition 4.4 (extended Penrose inequality for simple graphs) *Let* $H = (\mathsf{V}, \mathsf{E})$ *be a* simple *graph (i.e. no loops or multiple edges) equipped with complex edge weights* $\boldsymbol{w} = \{w_e\}_{e \in \mathsf{E}}$. *Then, for any vertex* $x \in \mathsf{V}$, *we have*

$$
\begin{aligned}
|C_H(\boldsymbol{w})| \;&\le\; T_H(\boldsymbol{w}'') \prod_{e \in \mathsf{E} \setminus \mathsf{E}(x)} \max\{1, |1+w_e|\} && \text{(4.10a)} \\
&\le\; T_H(\boldsymbol{w}'') \, \frac{\Psi(H, \boldsymbol{w})^{(|\mathsf{V}|-1)/2}}{\prod\limits_{e \in \mathsf{E}(x)} \max\{1, |1+w_e|\}^{1/2}} && \text{(4.10b)} \\
&\le\; T_H(\widetilde{\boldsymbol{w}}) \, \Psi(H, \boldsymbol{w})^{(|\mathsf{V}|-1)/2} && \text{(4.10c)} \\
&\le\; T_H(|\boldsymbol{w}|) \, \Psi(H, \boldsymbol{w})^{(|\mathsf{V}|-1)/2} && \text{(4.10d)}
\end{aligned}
$$

where $\mathsf{E}(x)$ *denotes the set of edges incident on* x,

$$
w''_e \;=\; \begin{cases} |w_e| & \text{if } e \in \mathsf{E}(x) \\ \min\{|w_e|,\, |w_e|/|1+w_e|\} & \text{otherwise} \end{cases} \tag{4.11}
$$

$$
\widetilde{w}_e \;=\; \begin{cases} \min\{|w_e|,\, |w_e|/|1+w_e|^{1/2}\} & \text{if } e \in \mathsf{E}(x) \\ \min\{|w_e|,\, |w_e|/|1+w_e|\} & \text{otherwise} \end{cases} \tag{4.12}
$$

and

$$
\Psi(H, \boldsymbol{w}) \;=\; \max_{y \in \mathsf{V}} \prod_{e \ni y} \max\{1, |1+w_e|\} \;. \tag{4.13}
$$

It is worth observing that (4.10a) is a genuine improvement of (4.5a), because the product $\prod_{e \in \mathsf{E}(x)} \max\{1, |1+w_e|\}$ more than compensates the factors $w''_e/w'_e = \max\{1, |1+w_e|\}$ for the *subset* of edges in $\mathsf{E}(x)$ that happen to lie in any given spanning tree of H [that is, we have $T_H(\boldsymbol{w}'') \le T_H(\boldsymbol{w}') \prod_{e \in \mathsf{E}(x)} \max\{1, |1+w_e|\}$]. Similarly, (4.10c) is always an improvement of (4.5b), since $T_H(\widetilde{\boldsymbol{w}}) \le T_H(\boldsymbol{w}')\, \Psi(H, \boldsymbol{w})^{1/2}$.

The main change in Proposition 4.4 compared to Proposition 4.3 is that the power of $\Psi(H, \boldsymbol{w})$ is reduced from $|\mathsf{V}|/2$ to $(|\mathsf{V}|-1)/2$. It is perhaps rather surprising that such a small modification in this intermediate bound results in the significant improvement obtained in going from Theorem 1.2 to Theorem 1.3 — namely, reducing the growth from $\Psi(G, \boldsymbol{w})$ to its square root — but that is how things turn out. An explanation will be given in Section 6, after the proof of Theorem 1.3.

We do pay a price for this improvement: namely, the argument of T_H in (4.10c) is $\widetilde{\boldsymbol{w}}$ rather than $\boldsymbol{w}'$. This is what causes the bound of Theorem 1.3 to depend on $\widetilde{\Delta}(G, \boldsymbol{w})$ rather than the somewhat smaller $\Delta'(G, \boldsymbol{w})$. We do not know whether this can be improved.

The proof of Proposition 4.4 — unlike all the preceding results in this section — depends on a specific choice of the map $\mathbf{R}$, namely the one used by Penrose in his original paper [24]. Let us briefly recall Penrose's construction (see [13, 12] for more details). We assume that $H = (\mathsf{V}, \mathsf{E})$ is a *simple* graph, and we choose (arbitrarily) an ordering of the vertex set V by numbering the vertices $1, 2, \ldots, n$ (where $n = |\mathsf{V}|$). We consider the vertex 1 to be the root, and denote it by r. If $T \subseteq \mathsf{E}$ is the edge set of a spanning tree in H [that is, (V, T) is a tree], then for each $x \in \mathsf{V}$ we denote by $\mathrm{dist}_T(x)$ the graph-theoretic distance in the tree (V, T) from the root r to the vertex x. Given T, the vertex set V is thus partitioned into "generations", defined as the sets of vertices at a given distance from the root r.

The Penrose map $\mathbf{R}\colon\, T \mapsto \mathbf{R}(T)$ is then defined as follows. For any tree $T \subseteq \mathsf{E}$, the edge set $\mathbf{R}(T) \supseteq T$ is obtained from T by adjoining all edges $e \in \mathsf{E}$ that either

(a) connect two vertices in the same generation [i.e. at equal distance from the root r in the tree (V, T) — note that no such edge can belong to T], or

(b) connect a vertex x to a vertex x' in the preceding generation [i.e. with $\mathrm{dist}_T(x') = \mathrm{dist}_T(x) - 1$] that is higher-numbered than the parent of x [here the parent of x is the unique vertex y with $\mathrm{dist}_T(y) = \mathrm{dist}_T(x) - 1$ such that $xy \in T$].

It can be shown [24, 13, 12] that $\mathbf{R}$ is indeed a partitioning map in the sense that $\mathcal{C}$ is the disjoint union of Boolean intervals $[T,\, \mathbf{R}(T)]$. Furthermore, it follows immediately from this construction that $\mathbf{R}(T) \setminus T$ cannot contain any edge incident on the root r; that is, $\mathbf{R}(T) \setminus T \subseteq \mathsf{E} \setminus \mathsf{E}(r)$.[7]

[7] We remark that this would no longer be the case in a generalization to the Penrose construction to non-simple graphs. In such a generalization, we would also order the edges connecting each pair of vertices, and we would add to the definition of $\mathbf{R}(T)$ a third case:

(c) connect a vertex x to its parent y by any edge that is higher-numbered than the edge con-

With the fact $\mathbf{R}(T) \setminus T \subseteq \mathsf{E} \setminus \mathsf{E}(r)$ in hand, the proof of Proposition 4.4 is almost immediate:

Proof of Proposition 4.4. We choose an ordering in which the vertex $x \in \mathsf{V}$ appearing in the statement of the proposition is numbered 1, i.e. $r = x$. Now follow the proof of Proposition 4.3 through (4.9b). We have $(T \cap \mathsf{E}_+) \cup (\mathbf{R}(T) \setminus T) \subseteq (\mathsf{E} \setminus \mathsf{E}(x)) \cup (T \cap \mathsf{E}_+ \cap \mathsf{E}(x))$. The terms in $\mathsf{E} \setminus \mathsf{E}(x)$ can be bounded above by $\prod_{e \in \mathsf{E} \setminus \mathsf{E}(x)} \max\{1, |1 + w_e|\}$ analogously to what is done in (4.9c), and the terms in $T \cap \mathsf{E}_+ \cap \mathsf{E}(x)$ convert the argument of T_H from $w'_e = \min\{|w_e|, |w_e|/|1 + w_e|\}$ to w''_e. This proves (4.10a).

Then

$$\prod_{e \in \mathsf{E} \setminus \mathsf{E}(x)} \max\{1, |1 + w_e|\} \;=\; \frac{\prod\limits_{y \in \mathsf{V} \setminus x} \prod\limits_{e \in \mathsf{E}(y)} \max\{1, |1 + w_e|\}^{1/2}}{\prod\limits_{e \in \mathsf{E}(x)} \max\{1, |1 + w_e|\}^{1/2}} \tag{4.14a}$$

$$\leq\; \frac{\Psi(H, \boldsymbol{w})^{(|\mathsf{V}|-1)/2}}{\prod\limits_{e \in \mathsf{E}(x)} \max\{1, |1 + w_e|\}^{1/2}} \tag{4.14b}$$

since the numerator of (4.14a) counts every edge in $\mathsf{E} \setminus \mathsf{E}(x)$ twice and every edge in $\mathsf{E}(x)$ once; so (4.10b) follows from (4.10a). Moreover, by definitions (4.11), (4.12) and (4.2) we have

$$\frac{T_H(\boldsymbol{w}'')}{\prod\limits_{e \in \mathsf{E}(x)} \max\{1, |1 + w_e|\}]^{1/2}} \;\leq\; T_H(\widetilde{\boldsymbol{w}}) \,, \tag{4.15}$$

so (4.10c) follows from (4.10b). Finally, (4.10d) is a trivial weakening of (4.10c). $\square$

5 Bounds on connected m-edge subgraphs containing a specified vertex

Let $G = (V, E)$ be a graph equipped with edge weights $\boldsymbol{w} = \{w_e\}_{e \in E}$. Let us define the weighted sum over connected subgraphs $G' = (V', E') \subseteq G$ that contain

---

necting x to y in T.

We would then no longer be able to guarantee that $\mathbf{R}(T) \setminus T$ contains no edges incident on the root r; rather, we could assert only that $\mathbf{R}(T) \setminus T$ cannot contain any edge incident on the root r that is the *lowest-numbered* among its set of parallel edges.

the specified vertex x and have exactly m edges:

$$c_m(x; G, \boldsymbol{w}) \;=\; \sum_{\substack{G'=(V',E')\subseteq G \\ G' \text{ connected} \\ V' \ni x \\ |E'|=m}} \prod_{e\in E'} |w_e| \;. \tag{5.1}$$

We will abbreviate $c_m(x; G, \boldsymbol{w})$ to $c_m(x)$ when it is obvious which weighted graph $(G, \boldsymbol{w})$ we are referring to. We then have the following sharp bound on $c_m(x)$ in terms of the maximum weighted degree

$$\Delta(G, \boldsymbol{w}) \;=\; \max_{y\in V} \sum_{e\ni y} |w_e| \;. \tag{5.2}$$

Proposition 5.1 [27, Proposition 4.5] *Let $G = (V, E)$ be a loopless graph equipped with edge weights $\boldsymbol{w} = \{w_e\}_{e\in E}$. Then for any $x \in V$, we have*

$$\begin{aligned} c_m(x) \;&\le\; \frac{(m+1)^{m-1}}{m!}\, \Delta(G, \boldsymbol{w})^m & \text{(5.3a)} \\ &\le\; e^m\, \Delta(G, \boldsymbol{w})^m \;. & \text{(5.3b)} \end{aligned}$$

[The e in (5.3b) denotes, of course, the base of the natural logarithms!]

See also [17, Section 6] for two alternate proofs of a slight generalization of Proposition 5.1, as well as examples showing that it is (in a certain sense) sharp.

We shall see in the next section that Proposition 5.1, combined with the results of Sections 2, 3 and 4, is sufficient to prove Theorem 1.2 (as well as a slightly weakened version of Theorem 1.3). In order to prove the full Theorem 1.3, we shall need a slight improvement of Proposition 5.1 in which the edges incident on the "root" vertex x are treated differently from the remaining edges. Let us write $E(x; G)$ for the set of all edges in G that are incident on x, and define the weighted degree at x by

$$d(x; G, \boldsymbol{w}) \;=\; \sum_{e\in E(x;G)} |w_e| \;. \tag{5.4}$$

We will abbreviate $E(x; G)$ to $E(x)$ when it is obvious which graph G we are referring to. We then have:

Proposition 5.2 *Let $G = (V, E)$ be a graph equipped with edge weights $\boldsymbol{w} = \{w_e\}_{e\in E}$ and let $x \in V$. Put $d = d(x; G, \boldsymbol{w})$ and $D = \Delta(G - x, \boldsymbol{w}|_{G-x})$. Then*

$$c_m(x) \;\le\; \frac{d\,(d + mD)^{m-1}}{m!} \;. \tag{5.5}$$

Since d and D are both bounded above by $\Delta(G, \boldsymbol{w})$, this is a genuine improvement of Proposition 5.1.

In the proof of Proposition 5.2 it will be convenient to employ the quantities

$$C(m,\kappa) \;=\; \begin{cases} \kappa(m+\kappa)^{m-1}/m! & \text{for } m \geq 1 \\ 1 & \text{for } m = 0 \end{cases} \tag{5.6}$$

defined for integer $m \geq 0$ and real κ. Then (5.5) can be rewritten in the form

$$c_m(x) \;\leq\; C(m, d/D)\, D^m \,. \tag{5.7}$$

One way of proving Proposition 5.2 is to use Lemma 5.3 below together with Proposition 5.1 applied to $G - x$. But it is instructive to give here an *ab initio* proof, by induction on m, that is similar to the first proof of Proposition 5.1 given in [17, Section 6]. It relies on the following properties of $C(m,\kappa)$:

(a) For each integer $m \geq 0$, $C(m,\kappa)$ is a polynomial of degree m in κ, with nonnegative coefficients. In particular, $C(m,\kappa)$ is an increasing function of κ for real $\kappa \geq 0$.

(b) Generating function: If $\mathsf{C}(z)$ solves the equation

$$\mathsf{C}(z) \;=\; e^{z\mathsf{C}(z)} \,, \tag{5.8}$$

then

$$\mathsf{C}(z)^\kappa \;=\; \sum_{m=0}^{\infty} C(m,\kappa)\, z^m \tag{5.9}$$

for all real κ; this follows from the Lagrange inversion formula. Moreover, the series (5.9) is absolutely convergent for $|z| \leq 1/e$ and satisfies $\mathsf{C}(1/e) = e$.

(c) For integer $k \geq 1$,

$$C(m,k) \;=\; \sum_{\substack{m_1,\ldots,m_k \geq 0 \\ m_1+\cdots+m_k = m}} \prod_{i=1}^{k} C(m_i, 1) \,. \tag{5.10}$$

This is an immediate consequence of (5.9).

(d) For all real κ and z,

$$C(m,\kappa) \;=\; \sum_{f=0}^{m} \frac{z^f}{f!}\, C(m-f, \kappa - z + f) \,. \tag{5.11}$$

See [17, eq. (6.7)].

For any subset $F \subseteq E$, we use the notation $w(F) = \prod_{e\in F} w_e$. We will need two further results from [17]:

Lemma 5.3 [17, Section 6, Facts 1 and 2] *Let $G = (V, E)$ be a graph equipped with nonnegative real edge weights $\boldsymbol{w} = \{w_e\}_{e\in E}$ and let $x \in V$. For each $F \subseteq E(x)$, let $Y^F = \{x_1^F, x_2^F, \ldots, x_{j(F)}^F\}$ be a labeling of the vertices of $V - x$ that are incident with edges in F. Then, for all $m \geq 1$,*

$$c_m(x; G, \boldsymbol{w}) \leq \sum_{\varnothing \neq F \subseteq E(x)} w(F) \sum_{\substack{m_1,\ldots,m_{j(F)} \geq 0 \\ m_1+\cdots+m_{j(F)} = m-|F|}} \prod_{i=1}^{j(F)} c_{m_i}(x_i^F; G - x, \boldsymbol{w}|_{G-x}) \,. \tag{5.12}$$

Please observe that $j(F) \leq |F|$; and if the graph G is simple, then $j(F) = |F|$. Please note also that Fact 2 as stated in [17] writes $c_{m_i}(x_i^F; G, \boldsymbol{w})$ in place of $c_{m_i}(x_i^F; G - x, \boldsymbol{w}|_{G-x})$ in (5.3), but the improved version stated here is an immediate consequence of the proof given there.

Lemma 5.4 [17, Lemma 6.2] *Let S be a set in which each element $e \in S$ is given a nonnegative real weight w_e. Then, for each integer $f \geq 0$, we have*

$$\sum_{\substack{F \subseteq S \\ |F| = f}} \prod_{e\in F} w_e \;\leq\; \frac{1}{f!} \left(\sum_{e\in S} w_e \right)^f \,. \tag{5.13}$$

Proof of Proposition 5.2. We will prove (5.5)/(5.7) by induction on m. The statement holds trivially when $m = 0$, so let us assume that $m \geq 1$. By Lemma 5.3,

$$\begin{aligned}
c_m(x) \;&\leq \sum_{\varnothing \neq F \subseteq E(x)} |w(F)| \sum_{\substack{m_1,\ldots,m_{j(F)} \geq 0 \\ m_1+\cdots+m_{j(F)} = m-|F|}} \prod_{i=1}^{j(F)} c_{m_i}(x_i^F; G - x, \boldsymbol{w}|_{G-x}) \\
&\leq \sum_{\varnothing \neq F \subseteq E(x)} |w(F)| \sum_{\substack{m_1,\ldots,m_{j(F)} \geq 0 \\ m_1+\cdots+m_{j(F)} = m-|F|}} \prod_{i=1}^{j(F)} C(m_i,\, 1)\; D^{m_i} \\
&= \sum_{\varnothing \neq F \subseteq E(x)} |w(F)|\; C(m - |F|,\, j(F))\; D^{m-|F|} \\
&\leq \sum_{\varnothing \neq F \subseteq E(x)} |w(F)|\; C(m - |F|,\, |F|)\; D^{m-|F|}
\end{aligned} \tag{5.14}$$

where the second line used the induction hypothesis (5.5) applied to the graph $G-x$ (note that $m_i < m$)[8] and the fact that $d(v; G-x, \boldsymbol{w}|_{G-x}) \le D$ for all $v \in V-x$, the third line used the identity (5.10), and the last line used $j(F) \le |F|$ and the fact that $C(m,k)$ is an increasing function of k. Using Lemma 5.4, we have

$$\begin{aligned} c_m(x) &\le D^m \sum_{f=1}^{m} \frac{(d/D)^f}{f!}\, C(m-f, f) \\ &= D^m \sum_{f=0}^{m} \frac{(d/D)^f}{f!}\, C(m-f, f) \\ &= C(m, d/D)\, D^m \,, \end{aligned} \tag{5.15}$$

where the second line used $C(m,0)=0$ for $m \ge 1$, and the last line used identity (5.11) with $\kappa = z = d/D$. This proves (5.7). $\square$

6 Proof of Theorems 1.2 and 1.3 and Lemma 1.4

We can now put together the results of the preceding sections to prove Theorems 1.2 and 1.3. At the end of this section we will also prove Lemma 1.4.

Proof of Theorem 1.2. We want to show that $Z_G(q, \boldsymbol{w}) \neq 0$ whenever $|q| \ge \mathcal{K}^*(\Psi(G,\boldsymbol{w}))\, \Delta'(G,\boldsymbol{w})$. We will do this by verifying the condition (3.4) for the polymer weights (2.3), which we recall are

$$\xi(S) \;=\; q^{-(|S|-1)} C_{G[S]}(\boldsymbol{w}) \qquad \text{for } |S| \ge 2\,. \tag{6.1}$$

By Proposition 4.3 [bound (4.5b)] applied to $H = G[S]$, we have

$$|C_{G[S]}(\boldsymbol{w})| \;\le\; T_{G[S]}(\boldsymbol{w}')\, \Psi(G[S], \boldsymbol{w})^{|S|/2} \tag{6.2a}$$

$$\le\; T_{G[S]}(\boldsymbol{w}')\, \Psi(G, \boldsymbol{w})^{|S|/2} \tag{6.2b}$$

where $\boldsymbol{w}'$ is defined by (4.6). Then by Proposition 5.1 applied to the weights $\boldsymbol{w}'$, and observing that the n-vertex trees are a subset of the connected graphs with $m = n-1$ edges, we have for any vertex $x \in V$

$$\sum_{\substack{S \ni x \\ |S|=n}} T_{G[S]}(\boldsymbol{w}') \;\le\; \frac{n^{n-1}}{n!}\, \Delta(G, \boldsymbol{w}')^{n-1} \tag{6.3a}$$

$$=\; \frac{n^{n-1}}{n!}\, \Delta'(G, \boldsymbol{w})^{n-1}\,. \tag{6.3b}$$

[8] Actually, we used here only the weakening (5.3a) of the inductive hypothesis (5.5), applied to $G-x$.

Therefore, the condition (3.4) for the weights (2.3)/(6.1) is verified as soon as

$$\inf_{\alpha>0}(e^\alpha-1)^{-1}\sum_{n=2}^{\infty} e^{\alpha n}\,|q|^{-(n-1)}\,\frac{n^{n-1}}{n!}\,\Delta'(G,\boldsymbol{w})^{n-1}\,\Psi(G,\boldsymbol{w})^{n/2}\;\le\;1\,. \tag{6.4}$$

If we set $L=|q|/\Delta'(G,\boldsymbol{w})$, this is precisely the inequality contained in the right-hand side of (1.7a). So $Z_G(q,\boldsymbol{w})\neq 0$ whenever $L\ge\mathcal{K}^*(\Psi(G,\boldsymbol{w}))$, i.e. whenever $|q|\ge\mathcal{K}^*(\Psi(G,\boldsymbol{w}))\,\Delta'(G,\boldsymbol{w})$.

The equivalence with (1.7b,c) and the inequality (1.7d) are proven in Lemma 6.1 and Corollary 6.2 below: see (6.13a–c). $\square$

Lemma 6.1 *For $\lambda\ge 0$ and $\beta>0$, define the function*

$$F_\lambda(\beta)\;=\;\min\left\{L:\;\inf_{\alpha>0}(e^\alpha-1)^{-1}\sum_{n=2}^{\infty}e^{\alpha n}\,L^{-(n-1)}\,\frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!}\;\le\;\beta\right\}\,. \tag{6.5}$$

Then:

(a) $F_\lambda(\beta)$ is an increasing function of λ and a decreasing function of β.

(b) $\beta F_\lambda(\beta)$ is an increasing function of both λ and β.

(c) $F_\lambda(\mu/\lambda)/\lambda$ is a decreasing function of both λ and μ (>0). In particular, $F_\lambda(\beta)/\lambda$ is a decreasing function of both λ and β.

(d) $\log F_\lambda(\beta)$ is a convex function of $\log\beta$.

(e) We have

$$F_\lambda(\beta)\;=\;\min_{1<y<1+\beta}\frac{\beta y^\lambda}{(1+\beta-y)\log y}\,. \tag{6.6}$$

(f) For $\lambda=0,1$ we have

$$F_0(\beta)\;=\;\frac{\beta}{1+\beta}\,W((1+\beta)e)\big/\left[W((1+\beta)e)\;-\;1\right]^2 \tag{6.7}$$

$$F_1(\beta)\;=\;\beta\,W\!\left(\frac{e}{1+\beta}\right)\Big/\left[1-W\!\left(\frac{e}{1+\beta}\right)\right]^2 \tag{6.8}$$

where W is the Lambert W function [11], i.e. the inverse function to $x\mapsto xe^x$.

(g) For $0\le\lambda\le\lambda'$ we have

$$F_\lambda(\beta)\;\le\;\frac{1+2\lambda}{1+2\lambda'}\,F_{\lambda'}\!\left(\frac{1+2\lambda}{1+2\lambda'}\,\beta\right)\,. \tag{6.9}$$

(h) For $0 \le \lambda \le 1$ we have

$$F_\lambda(\beta) \;\le\; 4\beta^{-1} + (1+2\lambda) \;. \tag{6.10}$$

Corollary 6.2 *The function*

$$\mathcal{K}^*(\psi) \;=\; \min\left\{ L\colon \; \inf_{\alpha>0} (e^\alpha - 1)^{-1} \sum_{n=2}^\infty e^{\alpha n} \, \psi^{n/2} \, L^{-(n-1)} \, \frac{n^{n-1}}{n!} \;\le\; 1 \right\} \tag{6.11}$$

defined for $\psi > 0$ satisfies

$$\mathcal{K}^*(\psi) \;=\; \psi^{1/2} F_1(\psi^{-1/2}) \tag{6.12}$$

and hence

$$\begin{aligned} \mathcal{K}^*(\psi) \;&=\; \min_{1<y<1+\psi^{-1/2}} \frac{y}{(1+\psi^{-1/2}-y)\log y} && \text{(6.13a)} \\ &=\; W\!\left(\frac{e}{1+\psi^{-1/2}}\right) \bigg/ \left[1 - W\!\left(\frac{e}{1+\psi^{-1/2}}\right)\right]^2 && \text{(6.13b)} \\ &\le\; 4\psi + 3\psi^{1/2} \;. && \text{(6.13c)} \end{aligned}$$

Only parts (e,f,h) of Lemma 6.1 will actually be used in the proofs of Theorems 1.2 and 1.3. But we think it worthwhile to collect here some additional properties of the function $F_\lambda(\beta)$: some of these will be invoked in the Discussion after the proof of Theorem 1.3, while others may end up playing a role in future work.

Proof of Lemma 6.1. (a) It is immediate from the definition (6.5) that $F_\lambda(\beta)$ is increasing in λ and decreasing in β.

(b) The change of variables $L' = \beta L$ in (6.5) shows that

$$\beta F_\lambda(\beta) \;=\; \min\left\{ L'\colon \; \inf_{\alpha>0} (e^\alpha - 1)^{-1} \sum_{n=2}^\infty e^{\alpha n} \, (L')^{-(n-1)} \, \beta^{n-2} \, \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \;\le\; 1 \right\} \tag{6.14}$$

is increasing in both λ and β.

(c) The change of variables $L'' = L/\lambda$ in (6.5) shows that

$$\frac{F_\lambda(\mu/\lambda)}{\lambda} \;=\; \min\left\{ L''\colon \; \inf_{\alpha>0} (e^\alpha - 1)^{-1} \sum_{n=2}^\infty e^{\alpha n} \, (L'')^{-(n-1)} \, \frac{[\lambda^{-1}+(n-1)]^{n-2}}{(n-1)!} \;\le\; \mu \right\} \tag{6.15}$$

is decreasing in both λ and μ.

(d) Suppose that we have triplets (α_i, L_i, β_i) satisfying

$$\sum_{n=2}^\infty e^{\alpha_i n} \, L_i^{-(n-1)} \, \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \;\le\; \beta_i \, (e^{\alpha_i} - 1) \tag{6.16}$$

for $i = 1, 2$. Now let $\kappa \in [0, 1]$ and define

$$\bar{\alpha} \;=\; \kappa\alpha_1 + (1-\kappa)\alpha_2 \tag{6.17a}$$

$$\bar{L} \;=\; L_1^{\kappa} L_2^{1-\kappa} \tag{6.17b}$$

$$\bar{\beta} \;=\; \beta_1^{\kappa} \beta_2^{1-\kappa} \tag{6.17c}$$

Then Hölder's inequality with $p = 1/\kappa$ and $q = 1/(1-\kappa)$ yields

$$\sum_{n=2}^{\infty} e^{\bar{\alpha} n} \, \bar{L}^{-(n-1)} \, \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \;\le\; \bar{\beta}\, (e^{\alpha_1}-1)^{\kappa}\, (e^{\alpha_2}-1)^{1-\kappa} \,. \tag{6.18}$$

And since the function $\alpha \mapsto \log(e^{\alpha}-1)$ is concave on $(0, \infty)$, we have $(e^{\alpha_1}-1)^{\kappa}(e^{\alpha_2}-1)^{1-\kappa} \le e^{\bar{\alpha}} - 1$. This proves (d).

(e) The proof that (6.5) is equivalent to (6.6) will be modelled on an argument of Borgs [9, eq. (4.22) ff.], who proved a related result.

Note first that $c \mapsto ce^{-c}$ maps the interval $[0, 1]$ strictly monotonically onto the interval $[0, 1/e]$; and recall [31, p. 28] that its inverse map is the tree function

$$T(x) \;=\; \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \, x^n \;, \tag{6.19}$$

which is convergent and monotonically increasing for $0 \le x \le 1/e$ and satisfies $T(ce^{-c}) = c$ for $0 \le c \le 1$. Moreover, it is well known (see e.g. [11, eq. (2.36)]) that for all real $\kappa > 0$ one has [cf. (5.9)]

$$\left(\frac{T(z)}{z}\right)^{\kappa} \;=\; \sum_{m=0}^{\infty} \frac{\kappa\,(m+\kappa)^{m-1}}{m!} \, z^m \tag{6.20}$$

(this is an easy consequence of the Lagrange inversion formula). Writing for convenience $U(z) = T(z)/z$, we therefore have

$$\sum_{n=1}^{\infty} \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \, z^n \;=\; z\, U(\lambda z)^{1/\lambda} \tag{6.21}$$

for all real $\lambda > 0$.

The inequality on the right-hand side of (6.5) is then equivalent to the statement that $\lambda e^{\alpha}/L \le 1/e$ (otherwise the sum would be divergent) and

$$e^{\alpha}\, U(\lambda e^{\alpha}/L)^{1/\lambda} \;-\; e^{\alpha} \;\le\; \beta(e^{\alpha}-1) \,. \tag{6.22}$$

Eliminating L in favor of a new variable c defined by $\lambda e^{\alpha}/L = ce^{-c}$ with $0 \le c \le 1$, and using the fact that $U(ce^{-c}) = e^c$, we see that the inequality on the right-hand side of (6.5) is equivalent to

$$c \;\le\; \min\Big\{1,\, \lambda \log[1+\beta(1-e^{-\alpha})]\Big\} \,. \tag{6.23}$$

Since $L = \lambda e^{\alpha}/(ce^{-c})$, and ce^{-c} increases monotonically with c for $0 \le c \le 1$, we deduce that (6.23) is equivalent to

$$L \;\ge\; \begin{cases} \dfrac{e^{\alpha}\,[1+\beta(1-e^{-\alpha})]^{\lambda}}{\log[1+\beta(1-e^{-\alpha})]} & \text{if } \beta(1-e^{-\alpha}) \le e^{1/\lambda}-1 \\[4mm] \lambda e^{\alpha+1} & \text{if } \beta(1-e^{-\alpha}) \ge e^{1/\lambda}-1 \end{cases} \tag{6.24}$$

Changing variables from α to $y = 1+\beta(1-e^{-\alpha})$, we can rewrite this as

$$L \;\ge\; \begin{cases} \dfrac{\beta y^{\lambda}}{(1+\beta-y)\log y} & \text{if } 1 < y < \min(e^{1/\lambda}, 1+\beta) \\[4mm] \dfrac{\lambda\beta e}{1+\beta-y} & \text{if } e^{1/\lambda} \le y < 1+\beta \end{cases} \tag{6.25}$$

Now we can optimize over y: the minimum will always be found in the interval $1 < y \le e^{1/\lambda}$, so we have

$$F_{\lambda}(\beta) \;=\; \min_{1<y<\min(e^{1/\lambda},1+\beta)} \frac{\beta y^{\lambda}}{(1+\beta-y)\log y} \;=\; \min_{1<y<1+\beta} \frac{\beta y^{\lambda}}{(1+\beta-y)\log y}\,, \tag{6.26}$$

where the final equality results from the fact that $y^{\lambda}/[(1+\beta-y)\log y]$ is increasing for $e^{1/\lambda} \le y < 1+\beta$. This proves the equivalence of (6.5) with (6.6) for $\lambda > 0$; and the case $\lambda = 0$ follows by taking limits (or by an easy direct proof).

(f) For $\lambda = 0$, simple calculus shows that the minimum in (6.6) is attained at $y = (1+\beta)/W((1+\beta)e)$, so that $F_0(\beta)$ is given by (6.7). Likewise, for $\lambda = 1$, simple calculus shows that the minimum in (6.6) is attained at $y = (1+\beta)\,W(e/(1+\beta))$, so that $F_1(\beta)$ is given by (6.8).

(g) To prove the comparison inequality (6.9), it suffices to observe that whenever $0 \le \lambda \le \lambda'$ and $n \ge 2$ we have

$$\left(\frac{1+(n-1)\lambda}{1+(n-1)\lambda'}\right)^{n-2} \;\le\; \left(\frac{1+2\lambda}{1+2\lambda'}\right)^{n-2} \tag{6.27}$$

(just consider $n = 2$ and $n \ge 3$ separately). Inserting this into the definition (6.5) yields (6.9).

(h) To prove the upper bound (6.10), write $y = 1+x$ in (6.6) and use the inequalities

$$\frac{1}{\log(1+x)} \;\le\; \frac{1}{x} + \frac{1}{2} \tag{6.28}$$

$$(1+x)^{\lambda} \;\le\; 1+\lambda x \tag{6.29}$$

which are valid for all $x > 0$ and $0 \le \lambda \le 1$.[9] Therefore,

$$\frac{\beta y^\lambda}{(1+\beta-y)\log y} \;\le\; \frac{\beta(1+\lambda x)\left(\frac{1}{x}+\frac{1}{2}\right)}{\beta - x}\;. \tag{6.30}$$

The latter function is minimized at $x = (-2+\sqrt{4+(2+4\lambda)\beta+2\lambda\beta^2})/[1+(2+\beta)\lambda] \in (0,\beta)$, with minimum value

$$\frac{1}{2} + \lambda + \frac{2}{\beta} + \frac{2}{\beta}\sqrt{(1+\beta/2)(1+\lambda\beta)}\;. \tag{6.31}$$

This, in turn, is bounded above by $4\beta^{-1}+(1+2\lambda)$ on the entire interval $0 < \beta < \infty$.[10] [Alternatively, it suffices to make this proof for $\lambda = 1$ and then invoke (6.9) to deduce the result for $0 \le \lambda < 1$.] $\square$

Proof of Corollary 6.2. An easy calculation proves the formula (6.12) relating $\mathcal{K}^*(\psi)$ to $F_1(\beta)$, so that (6.13a–c) follow from (6.6), (6.8) and (6.10). $\square$

Remarks. 1. The proof of Lemma 6.1(e) becomes a bit simpler for $\beta \le e^{1/\lambda}-1$, since we then always have $\beta(1-e^{-\alpha}) \le e^{1/\lambda}-1$ and hence we need not worry about the second case in (6.24) and (6.25). This simplification applies in particular when $\lambda \le 1$ and $\beta \le 1$, which covers what is needed in the proofs of both Theorem 1.2 ($\lambda = 1$, $\beta = \psi^{-1/2} \le 1$) and Theorem 1.3 ($0 < \lambda \le 1$, $\beta = 1$).

2. We can compute the small-β asymptotics of $F_\lambda(\beta)$ by expanding (6.6) in powers of $y-1$: the minimum is located at

$$y \;=\; 1+\frac{1}{2}\beta - \frac{1+2\lambda}{16}\beta^2 + \frac{5+12\lambda}{192}\beta^3 - \frac{43+122\lambda+12\lambda^2-24\lambda^3}{3072}\beta^4 + \ldots \tag{6.32}$$

and we have

$$F_\lambda(\beta) \;=\; 4\beta^{-1} \;+\; (1+2\lambda) \;-\; \frac{7+12\lambda-12\lambda^2}{48}\beta \;+\; \frac{11+26\lambda-12\lambda^2-8\lambda^3}{192}\beta^2 \;+\; \ldots\;. \tag{6.33}$$

For $\lambda = 0, 1$ an alternate method is to expand (6.7)/(6.8): we obtain

$$F_0(\beta) \;=\; 4\beta^{-1} \;+\; 1 \;-\; \frac{7}{48}\beta \;+\; \frac{11}{192}\beta^2 \;-\; \frac{443}{15360}\beta^3 \;+\; \frac{607}{36864}\beta^4 \;-\; \ldots \tag{6.34}$$

$$F_1(\beta) \;=\; 4\beta^{-1} \;+\; 3 \;-\; \frac{7}{48}\beta \;+\; \frac{17}{192}\beta^2 \;-\; \frac{923}{15360}\beta^3 \;+\; \frac{8113}{184320}\beta^4 \;-\; \ldots \tag{6.35}$$

---

[9] Proof of (6.28): Write $t = \log(1+x) > 0$; then (6.28) states that $1/t \le 1/(e^t-1)+1/2$. This is trivially true for $t \ge 2$; and for $0 < t < 2$ it is equivalent to $e^t - 1 \le t/(1-t/2)$, which is obvious from the Taylor series.

[10] Proof: We have

$$\sqrt{(1+c_1\beta)(1+c_2\beta)} \;\le\; 1 \;+\; \frac{c_1+c_2}{2}\beta$$

for all $c_1, c_2, \beta \ge 0$, as is easily seen by squaring both sides and using the arithmetic-geometric-mean inequality $\sqrt{c_1c_2} \le (c_1+c_2)/2$.

Therefore, the large-ψ asymptotics of $\mathcal{K}^*(\psi) = \psi^{1/2} F_1(\psi^{-1/2})$ is

$$\mathcal{K}^*(\psi) \;=\; 4\psi + 3\psi^{1/2} - \frac{7}{48} + \frac{17}{192}\psi^{-1/2} - \frac{923}{15360}\psi^{-1} + \frac{8113}{184320}\psi^{-3/2} - \ldots \; . \tag{6.36}$$

3. We conjecture (based on plots of F_1 and its derivatives) that $F_1(\beta)$ is a *completely monotone* function of β on $(0,\infty)$, i.e. $(-1)^k\, d^k F_1(\beta)/d\beta^k \geq 0$ for all $\beta > 0$ and all integers $k \geq 0$.[11] Indeed, it seems that $G_1(\beta) = F_1(\beta) - 4/\beta$ is completely monotone, which is stronger.

Even more strongly, we conjecture (based on computations for $\operatorname{Im}\beta > 0$) that $G_\lambda(\beta) = F_\lambda(\beta) - 4/\beta$ is a *Stieltjes function* for $\lambda = 0$ and $\lambda = 1$, i.e. it can be written in the form

$$f(\beta) \;=\; C \,+ \int\limits_{[0,\infty)} \frac{d\rho(t)}{\beta + t} \tag{6.37}$$

where $C \geq 0$ and ρ is a positive measure on $[0,\infty)$.[12] It is even possible that this holds also for $0 < \lambda < 1$. $\square$

Proof of Theorem 1.3. We modify the proof of Theorem 1.2 by using Propositions 4.4 and 5.2 in place of Propositions 4.3 and 5.1, respectively. By Proposition 4.4 [bound (4.10c)], for any $x \in V$ and $S \subseteq V$ with $x \in S$, we have

$$|C_{G[S]}(\boldsymbol{w})| \;\leq\; T_{G[S]}(\widetilde{\boldsymbol{w}})\, \Psi(G[S], \boldsymbol{w})^{(|S|-1)/2} \tag{6.38a}$$

$$\leq\; T_{G[S]}(\widetilde{\boldsymbol{w}})\, \Psi(G, \boldsymbol{w})^{(|S|-1)/2} \tag{6.38b}$$

where $\widetilde{\boldsymbol{w}}$ is defined by (4.12). We may apply Proposition 5.2 to deduce that

$$\sum_{\substack{S \ni x \\ |S| = n}} T_{G[S]}(\widetilde{\boldsymbol{w}}) \;\leq\; d(x; G, \widetilde{\boldsymbol{w}})\, \frac{[d(x; G, \widetilde{\boldsymbol{w}}) \,+\, (n-1)\Delta(G - x, \widetilde{\boldsymbol{w}}|_{G-x})]^{n-2}}{(n-1)!} \;. \tag{6.39}$$

By the definition (4.12) of $\widetilde{\boldsymbol{w}}$ we have

$$d(x; G, \widetilde{\boldsymbol{w}}) \;\leq\; \max_{x \in V} \sum_{e \ni x} \min\left\{ |w_e|, \frac{|w_e|}{|1 + w_e|^{1/2}} \right\} \;\doteq\; \widetilde{\Delta}(G, \boldsymbol{w}) \tag{6.40a}$$

$$\Delta(G - x, \widetilde{\boldsymbol{w}}|_{G-x}) \;\leq\; \Delta(G, \boldsymbol{w}') \;=\; \Delta'(G, \boldsymbol{w}) \tag{6.40b}$$

[11] See e.g. [32] for the theory of completely monotone functions on $(0,\infty)$ — in particular the Bernstein–Hausdorff–Widder theorem, which states that a function is completely monotone on $(0,\infty)$ if and only if it is the Laplace transform of a positive measure supported on $[0,\infty)$.

[12] More information on Stieltjes functions can be found in [32] [1, pp. 126–128] [2, 3, 4, 30] and the references cited therein. In order to test numerically the Stieltjes property for $G_0(\beta)$ and $G_1(\beta)$, we have used the complex-analysis characterization: a function $f\colon (0,\infty) \to \mathbb{R}$ is Stieltjes if and only if it is the restriction to $(0,\infty)$ of an analytic function on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ satisfying $f(z) \geq 0$ for $z > 0$ and $\operatorname{Im} f(z) \leq 0$ for $\operatorname{Im} z > 0$. See e.g. [1, p. 127] or [3].

Hence

$$
\begin{aligned}
\sum_{\substack{S \ni x \\ |S|=n}} T_{G[S]}(\boldsymbol{w}'') &\leq \widetilde{\Delta}(G,\boldsymbol{w}) \, \frac{[\widetilde{\Delta}(G,\boldsymbol{w}) + (n-1)\Delta'(G,\boldsymbol{w})]^{n-2}}{(n-1)!} \\
&= \widetilde{\Delta}(G,\boldsymbol{w})^{n-1} \, \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \qquad (6.41)
\end{aligned}
$$

where $\lambda = \Delta'(G,\boldsymbol{w})/\widetilde{\Delta}(G,\boldsymbol{w})$. Therefore, the condition (3.4) for the weights (2.3)/(6.1) is verified as soon as

$$
\inf_{\alpha>0} (e^{\alpha}-1)^{-1} \sum_{n\geq 2} e^{\alpha n} \, [|q|^{-1}\Psi(G,\boldsymbol{w})^{1/2}\widetilde{\Delta}(G,\boldsymbol{w})]^{n-1} \, \frac{[1+(n-1)\lambda]^{n-2}}{(n-1)!} \;\leq\; 1 \, . \qquad (6.42)
$$

If we set $L = |q|\Psi(G,\boldsymbol{w})^{-1/2}\widetilde{\Delta}(G,\boldsymbol{w})^{-1}$, this is precisely the inequality contained in the right-hand side of (1.10a), or equivalently in the right-hand side of (6.5) with $\beta = 1$. So $Z_G(q,\boldsymbol{w}) \neq 0$ whenever $L \geq K_\lambda^* = F_\lambda(1)$, i.e. whenever $|q| \geq K_\lambda^* \, \Psi(G,\boldsymbol{w})^{1/2} \, \widetilde{\Delta}(G,\boldsymbol{w})$. The equivalence with (1.10b) and the inequality (1.10c) then follow from Lemma 6.1. $\square$

Remark. A weaker version of Theorem 1.3, in which K_λ^* and $\widetilde{\Delta}(G,\boldsymbol{w})$ are replaced by K_1^* and $\Delta(G,\boldsymbol{w})$, respectively, can be proven by the same argument but using Proposition 5.1 in place of Proposition 5.2; moreover, we do not need to use (4.10c) but only (4.10d). $\square$

Discussion. 1. We can now understand why the apparently minor improvement from Proposition 4.3 to Proposition 4.4 — changing a factor $\Psi(G,\boldsymbol{w})^{n/2}$ to $\Psi(G,\boldsymbol{w})^{(n-1)/2}$ — leads to the significant improvement (in most cases) of the final bound from Theorem 1.2 to Theorem 1.3, namely, replacing a growth $\sim \Psi(G,\boldsymbol{w})$ by $\sim \Psi(G,\boldsymbol{w})^{1/2}$. Indeed, we can see using Lemma 6.1 that whenever we have a bound of the form

$$
\sum_{\substack{S \ni x \\ |S|=n}} |C_{G[S]}(\boldsymbol{w})| \;\leq\; \frac{n^{n-1}}{n!} \, \Delta^{n-1} \, \Psi^{(n-1)/2+b} \, , \qquad (6.43)
$$

we will obtain a bound on roots of $Z_G(q,\boldsymbol{w})$ of the form

$$
|q| \;<\; \Delta \, \Psi^{1/2} \, F_1(\Psi^{-b}) \, . \qquad (6.44)
$$

Proposition 4.3 and Theorem 1.2 correspond to $b = 1/2$, while Proposition 4.4 and Theorem 1.3 correspond to $b = 0$.

2. Let us compare the bounds provided by Theorems 1.2 and 1.3:

$$
\text{Theorem 1.2:} \qquad \mathcal{K}^*(\Psi(G,\boldsymbol{w})) \, \Delta'(G,\boldsymbol{w}) \qquad (6.45a)
$$

$$
\text{Theorem 1.3:} \qquad K_\lambda^* \, \Psi(G,\boldsymbol{w})^{1/2} \, \widetilde{\Delta}(G,\boldsymbol{w}) \qquad (6.45b)
$$

where $\lambda = \Delta'(G, \boldsymbol{w})/\widetilde{\Delta}(G, \boldsymbol{w}) \in (0, 1]$. Their ratio is therefore

$$\frac{\text{Theorem 1.3}}{\text{Theorem 1.2}} \;=\; \frac{K_\lambda^* \, \Psi(G, \boldsymbol{w})^{1/2}}{\lambda \, \mathcal{K}^*(\Psi(G, \boldsymbol{w}))} \;=\; \frac{F_\lambda(1)}{\lambda \, F_1(\Psi(G, \boldsymbol{w})^{-1/2})} \,. \tag{6.46}$$

Now, it is not difficult to see that $\widetilde{\Delta}(G, \boldsymbol{w}) \le \Delta'(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w})^{1/2}$, or in other words $\Psi(G, \boldsymbol{w})^{-1/2} \le \lambda$.[13] Since $F_1(\beta)$ is a decreasing function of β by Lemma 6.1(a), we have $F_1(\Psi(G, \boldsymbol{w})^{-1/2}) \ge F_1(\lambda)$ and hence

$$\frac{\text{Theorem 1.3}}{\text{Theorem 1.2}} \;\le\; \frac{F_\lambda(1)}{\lambda \, F_1(\lambda)} \;\equiv\; g(\lambda) \,. \tag{6.47}$$

Both $F_\lambda(1)$ and $\lambda \, F_1(\lambda)$ are increasing functions of λ by Lemma 6.1(a,b), but their ratio $g(\lambda)$ does not have any obvious monotonicity. *Numerically* we find that $g(\lambda)$ decreases from the value $K_0^*/4 \approx 1.223222$ at $\lambda = 0$ to a minimum value ≈ 0.930714 at $\lambda \approx 3.70249$, and then increases to 1 as $\lambda \to \infty$. We have not succeeded in *proving* that $g(\lambda) \le g(0)$ for $\lambda \in [0, 1]$, but if is true we can conclude that Theorem 1.3 is never more than a factor ≈ 1.223222 worse than Theorem 1.2. And in any case we have

$$g(\lambda) \;\le\; \frac{F_1(1)}{\lim\limits_{\lambda \to 0} \lambda \, F_1(\lambda)} \;=\; \frac{K_1^*}{4} \;\approx\; 1.726913 \quad \text{for } \lambda \in [0, 1] \,. \tag{6.48}$$

We shall see in Examples 7.1 and 7.2 that Theorem 1.3 can indeed be up to a factor ≈ 1.223222 worse than Theorem 1.2.

3. It is curious that the bound of Theorem 1.3 is not always better than that of Theorem 1.2, despite using better "ingredients" in its proof: namely, the bound (4.10c) from Proposition 4.4 always beats the bound (4.5b) from Proposition 4.3; and Proposition 5.2 is always better than Proposition 5.1. How is it that the final result can sometimes be worse?

The explanation is that the ratio of the bounds (4.10c) and (4.5b),

$$\frac{(4.10\text{c})}{(4.5\text{b})} \;=\; \frac{T_H(\widetilde{\boldsymbol{w}}) \, \Psi(H, \boldsymbol{w})^{(|\mathsf{V}|-1)/2}}{T_H(\boldsymbol{w}') \, \Psi(H, \boldsymbol{w})^{|\mathsf{V}|/2}} \;=\; \frac{T_H(\widetilde{\boldsymbol{w}})}{T_H(\boldsymbol{w}')} \, \Psi(H, \boldsymbol{w})^{-1/2} \,, \tag{6.49}$$

is the product of a "bad" factor $T_H(\widetilde{\boldsymbol{w}})/T_H(\boldsymbol{w}')$ and a "good" factor $\Psi(H, \boldsymbol{w})^{-1/2}$. Now, the "bad" factor $T_H(\widetilde{\boldsymbol{w}})/T_H(\boldsymbol{w}')$ is always bounded by $\Psi(H, \boldsymbol{w})^{1/2}$ — which is

---

[13] Proof. For each edge e we have

$$\begin{aligned} \min\left\{|w_e|, \frac{|w_e|}{|1+w_e|^{1/2}}\right\} \;&=\; \min\left\{|w_e|, \frac{|w_e|}{|1+w_e|}\right\} \times \max\{1, |1+w_e|\} \\ &\le\; \min\left\{|w_e|, \frac{|w_e|}{|1+w_e|}\right\} \times \Psi(G, \boldsymbol{w})^{1/2} \,. \end{aligned}$$

Summing this over $e \ni x$ and taking the maximum over $x \in V$, we obtain the desired inequality.

why (4.10c) is always better than (4.5b) — so it follows that

$$\frac{\sum\limits_{S\ni x,\, |S|=n} T_{G[S]}(\widetilde{\boldsymbol{w}})}{\sum\limits_{S\ni x,\, |S|=n} T_{G[S]}(\boldsymbol{w}')} \;\le\; \Psi(G,\boldsymbol{w})^{1/2} \,. \tag{6.50}$$

But there is no guarantee that the *upper bounds* on the numerator and denominator of (6.50), obtained by applying Propositions 5.2 and 5.1, respectively [cf. (6.39) and (6.3)], will also have a ratio $\le \Psi(G,\boldsymbol{w})^{1/2}$. It is true that Proposition 5.2 is always better than Proposition 5.1 *for a given set of weights* $\boldsymbol{w}$, but it could well happen that Proposition 5.2 applied to the weights $\widetilde{\boldsymbol{w}}$ loses more than Proposition 5.1 applied to the weights $\boldsymbol{w}'$. And this is indeed what happens in some cases (see Examples 7.1 and 7.2).

It is, nevertheless, somewhat disconcerting that Theorem 1.3 is not always better than Theorem 1.2. It would be interesting to try to find a single natural bound that simultaneously improves Theorems 1.2 and 1.3.

4. Let us sketch how to obtain a bound that "interpolates" between Propositions 4.3 and 4.4, and thence between Theorems 1.2 and 1.3, indexed by a parameter $a \in [0,1]$. The first step is to notice that the proofs of Propositions 4.3 and 4.4 bound the summand of (4.9b), individually for each tree T, in two different ways: one is (4.9c), and the other is the analogous bound in which $w'_e = \min\{|w_e|,\, |w_e|/|1+w_e|\}$ is replaced by w''_e and the product is restricted to $\mathsf{E} \setminus \mathsf{E}(x)$. So, taking for each T the weighted geometric mean of these two bounds, we obtain

$$|C_H(\boldsymbol{w})| \;\le\; T_H(\boldsymbol{w}^{(a)})\, \Psi(H,\boldsymbol{w})^{(|\mathsf{V}|-a)/2} \tag{6.51}$$

where

$$w_e^{(a)} \;=\; \begin{cases} \min\{|w_e|,\, |w_e|/|1+w_e|^{1-a/2}\} & \text{if } e \in \mathsf{E}(x) \\ \min\{|w_e|,\, |w_e|/|1+w_e|\} & \text{otherwise} \end{cases} \tag{6.52}$$

Imitating the proof of Theorem 1.3, we then obtain the bound

$$|q| \;<\; \Delta'_a(G,\boldsymbol{w})\, \Psi(G,\boldsymbol{w})^{1/2}\, F_{\lambda_a}(\Psi(G,\boldsymbol{w})^{-(1-a)/2}) \tag{6.53}$$

where

$$\Delta'_a(G,\boldsymbol{w}) \;=\; \Delta(G,\boldsymbol{w}^{(a)}) \;=\; \max_{x\in V} \sum_{e\ni x} \min\left\{ |w_e|,\, \frac{|w_e|}{|1+w_e|^{1-a/2}} \right\} \tag{6.54}$$

(so that $\Delta'_0 = \Delta'$ and $\Delta'_1 = \widetilde{\Delta}$) and $\lambda_a = \Delta'(G,\boldsymbol{w})/\Delta'_a(G,\boldsymbol{w}) \in (0,1]$. We then have

$$\frac{\text{bound (6.53)}}{\text{Theorem 1.2}} \;=\; \frac{F_{\lambda_a}(\Psi(G,\boldsymbol{w})^{-(1-a)/2})}{\lambda_a\, F_1(\Psi(G,\boldsymbol{w})^{-1/2})} \tag{6.55a}$$

$$\le\; \frac{F_{\Psi(G,\boldsymbol{w})^{-a/2}}(\Psi(G,\boldsymbol{w})^{-(1-a)/2})}{\Psi(G,\boldsymbol{w})^{-a/2}\, F_1(\Psi(G,\boldsymbol{w})^{-1/2})} \tag{6.55b}$$

where the inequality uses $\lambda_a \geq \Psi(G,\boldsymbol{w})^{-a/2}$ together wirh the fact that $F_\lambda(\beta)/\lambda$ is a decreasing function of λ by Lemma 6.1(c). Now, the *upper bound* (6.55b) is an increasing function of $a \in [0,1]$ for all $\Psi(G,\boldsymbol{w}) \geq 1$: this follows from Lemma 6.1(c) with $\lambda = \Psi(G,\boldsymbol{w})^{-a/2}$ and $\mu = \Psi(G,\boldsymbol{w})^{-1/2}$. But it is not clear what are the possible behaviors of the *true* bound (6.55a). For $G = K_2$ (see Example 7.1 below), we have $\lambda_a = \Psi(G,\boldsymbol{w})^{-a/2}$, so that (6.55b) is an *equality*, and the best bound is $a = 0$ (i.e. Theorem 1.2). On the other hand, in cases like Examples 7.5 and 7.6, the best bound is clearly $a = 1$ (i.e. Theorem 1.3) because it has the smallest power of $\Psi(G,\boldsymbol{w})$. We do not know whether there are examples in which some value $a \in (0,1)$ might be optimal. $\square$

Finally, let us prove Lemma 1.4 concerning the behavior of $\Psi(G,\boldsymbol{w})$ and $\Delta'(G,\boldsymbol{w})$ under parallel reduction:

PROOF OF LEMMA 1.4. Inequality (1.11) follows immediately from the fact that $(1+w_1)(1+w_2) = 1+w_3$. To prove (1.12), let us consider the following cases:

Case 1: $|1+w_1| \leq 1$ and $|1+w_2| \leq 1$. Then $\min\left\{|w_i|, \frac{|w_i|}{|1+w_i|}\right\} = |w_i|$ for $1 \leq i \leq 3$, so we just have to prove that $|w_3| \leq |w_1| + |w_2|$. Since $w_3 = w_1 + w_2 + w_1 w_2$, we have

$$\begin{aligned} |w_3| \;&=\; |w_1 + w_2 + w_1 w_2| \;=\; |w_1 + w_2(1+w_1)| \;\leq\; |w_1| + |w_2(1+w_1)| \\ &=\; |w_1| + |w_2|\,|1+w_1| \;\leq\; |w_1| + |w_2| \end{aligned} \tag{6.56}$$

since $|1+w_1| \leq 1$.

Case 2: $|1+w_1| \geq 1$ and $|1+w_2| \geq 1$. Then $\min\left\{|w_i|, \frac{|w_i|}{|1+w_i|}\right\} = \frac{|w_i|}{|1+w_i|}$ for $1 \leq i \leq 3$. Let $w_i' = -\frac{w_i}{1+w_i}$ for $1 \leq i \leq 3$, so that $1 + w_i' = (1+w_i)^{-1}$ for $1 \leq i \leq 3$ and hence $(1+w_1')(1+w_2') = 1 + w_3'$. Since $|1+w_1'| \leq 1$ and $|1+w_2'| \leq 1$, we may apply Case 1 to w_1', w_2', w_3' to deduce that $|w_3'| \leq |w_1'| + |w_2'|$, as required.

Case 3: $|1+w_1| \leq 1$, $|1+w_2| \geq 1$ and $|1+w_1|\,|1+w_2| \leq 1$. Then $\min\left\{|w_i|, \frac{|w_i|}{|1+w_i|}\right\} = |w_i|$ for $i \in \{1,3\}$, and $\min\left\{|w_2|, \frac{|w_2|}{|1+w_2|}\right\} = \frac{|w_2|}{|1+w_2|}$. By hypothesis we have $|1+w_1| \leq |1+w_2|^{-1}$. Hence

$$|w_3| \;=\; |w_1 + w_2(1+w_1)| \;\leq\; |w_1| + |w_2|\,|1+w_1| \;\leq\; |w_1| + \frac{|w_2|}{|1+w_2|}\,, \tag{6.57}$$

as required.

Case 4: $|1+w_1| \leq 1$, $|1+w_2| \geq 1$ and $|1+w_1|\,|1+w_2| \geq 1$. Then $\min\left\{|w_1|, \frac{|w_1|}{|1+w_1|}\right\} = |w_1|$, and $\min\left\{|w_i|, \frac{|w_i|}{|1+w_i|}\right\} = \frac{|w_i|}{|1+w_i|}$ for $i \in \{2,3\}$. Let $w_i' = -\frac{w_i}{1+w_i}$ for $1 \leq i \leq 3$. Then $|1+w_1'| \geq 1$ and $|1+w_2'| \leq 1$ with $|1+w_1'|\,|1+w_2'| \leq 1$, so we may apply Case 3 (with indices 1 and 2 interchanged) to deduce that $|w_3'| \leq \frac{|w_1'|}{|1+w_1'|} + |w_2'| = |w_1| + |w_2'|$, as required. $\square$

Remark. We suspect that the transformation

$$w' \;=\; -\frac{w}{1+w} \tag{6.58}$$

employed in Cases 2 and 4, which satisfies $(1 + w') = (1 + w)^{-1}$ and hence preserves the parallel-connection law $(1+w_1)(1+w_2) = 1+w_3$, may have other applications in the study of the multivariate Tutte polynomial. This transformation is involutive (i.e. $w'' = w$), maps the complex antiferromagnetic regime $|1 + w| \le 1$ onto the complex ferromagnetic regime $|1+w'| \ge 1$ and vice versa, and maps the real antiferromagnetic regime $-1 \le w \le 0$ onto the real ferromagnetic regime $0 \le w' \le +\infty$ and vice versa. In the physicists' notation $w = e^J - 1$ where J is the Potts-model coupling, the transformation (6.58) takes the simple form $J' = -J$, which makes its properties obvious.

This line of reasoning also suggests that the quantity

$$\Delta'(G, \boldsymbol{w}) \;=\; \max_{x\in V} \sum_{e\ni x} \min\left\{|w_e|,\, \frac{|w_e|}{|1+w_e|}\right\} \;=\; \max_{x\in V} \sum_{e\ni x} \min\{|w_e|,\, |w'_e|\} \tag{6.59}$$

arising in Theorem 1.2, which we introduced simply because it arose naturally in our proof of Proposition 4.3, may also be "natural" in some more fundamental sense. □

7 Examples

In this section we examine some examples that shed light on the extent to which Theorems 1.2 and 1.3 are sharp or non-sharp. For each graph G, we attempt to compute or estimate the quantity

$$Q_{\max}(G, \boldsymbol{w}) \;=\; \max\{|q|\colon\; Z_G(q, \boldsymbol{w}) = 0\} \tag{7.1}$$

and compare it to the upper bounds given by Theorem 1.2 and Theorem 1.3. In what follows we abbreviate $\Delta'(G, \boldsymbol{w})$, $\widetilde{\Delta}(G, \boldsymbol{w})$, $\Psi(G, \boldsymbol{w})$, $Q_{\max}(G, \boldsymbol{w})$ by Δ', $\widetilde{\Delta}$, Ψ, $Q_{\max}$.

Example 7.1 Let $G = K_2$, where the single edge has weight w. Then $Z_{K_2}(q, w) = q(q+w)$, so that $Q_{\max} = |w|$. On the other hand, if $|1+w| \ge 1$ we have $\Delta' = |w|/|1+w|$, $\widetilde{\Delta} = |w|/|1+w|^{1/2}$, $\Psi = |1+w|$ and $\lambda = \Delta'/\widetilde{\Delta} = 1/|1+w|^{1/2}$. Theorem 1.2 gives the bound $|q| < \mathcal{K}^*(\Psi)\,\Delta'$, which behaves like $4|w|$ as $|w| \to \infty$, while Theorem 1.3 gives the bound $|q| < K^*_\lambda\,\Psi^{1/2}\,\widetilde{\Delta}$, which behaves like $K^*_0|w| \approx 4.892888|w|$ as $|w| \to \infty$. So Theorem 1.2 is off by a factor of 4 from the truth, while Theorem 1.3 is off by a factor of ≈ 4.892888 from the truth. In particular, Theorem 1.3 is worse than Theorem 1.2 by a factor tending to $K^*_0/4 \approx 1.223222$.

For the special case of $G = K_2$, the convergence conditions (6.4) and (6.42), which were used in the proofs of Theorems 1.2 and 1.3, respectively, become

$$\inf_{\alpha>0} (e^\alpha - 1)^{-1} e^{2\alpha} \, |q|^{-1} \, \Delta'(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w}) \;\; \leq \;\; 1 \tag{7.2}$$

$$\inf_{\alpha>0} (e^\alpha - 1)^{-1} e^{2\alpha} \, |q|^{-1} \, \widetilde{\Delta}(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w})^{1/2} \;\; \leq \;\; 1 \tag{7.3}$$

because the only polymer in the graph K_2 has size $n = 2$. Since $\Delta'(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w}) = \widetilde{\Delta}(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w})^{1/2} = |w|$, we have

$$(7.2) \iff (7.3) \iff |q| \geq 4|w| \, , \tag{7.4}$$

which differs from the truth $Q_{\max} = |w|$ by a factor of 4. We can understand this behavior as follows:

1) The lost factor of 4 comes from the fact that, for a polymer gas consisting of a single polymer S of cardinality $|S| = 2$, the Gruber–Kunz–Fernández–Procacci condition (Proposition 3.1) gives $\Xi \neq 0$ whenever $|\rho(S)| \leq 1/4$, whereas the truth is that $\Xi \neq 0$ whenever $|\rho(S)| < 1$.

2) Though the convergence condition (6.4) involves a sum $\sum_{n=2}^{\infty}$, the terms for $n > 2$ make a negligible contribution in the limit $|w| \to \infty$ because with $|q|$ of order $|w|$ we have

$$|q|^{-(n-1)} \, \Delta'(G, \boldsymbol{w})^{n-1} \, \Psi(G, \boldsymbol{w})^{n/2} \;=\; (|w|/|q|)^{n-1} \, |1 + w|^{-(\frac{n}{2}-1)} \;\to\; 0 \tag{7.5}$$

as $|w| \to \infty$ whenever $n > 2$. That is why Theorem 1.2 is off from the truth by the *same* factor 4 that we see in (7.4), despite the fact that its proof allows for arbitrarily large polymers that do not occur when $G = K_2$.

3) By contrast, in the convergence condition (6.42), the terms with $n > 2$ do *not* disappear in the limit $|w| \to \infty$ with $|q|$ of order $|w|$, because

$$[|q|^{-1} \, \widetilde{\Delta}(G, \boldsymbol{w}) \, \Psi(G, \boldsymbol{w})^{1/2}]^{n-1} \;=\; (|w|/|q|)^{n-1} \tag{7.6}$$

is of order 1 for all n. This is why Theorem 1.3 is off from the truth by *more* than the factor 4 that we see in (7.4); we lose an additional factor $K_0^*/4 \approx 1.223222$ by allowing for nonexistent large polymers. $\square$

Example 7.2 In *any* simple graph G with at least one edge, we can choose weights $\boldsymbol{w}$ such that Theorem 1.2 beats Theorem 1.3 by a factor arbitrarily close to $K_0^*/4 \approx 1.223222$. It suffices to take $w_e = w$ (with $|1 + w| \geq 1$) on all the edges of a nonempty matching, and $w_e = w_0$ on all other edges; then as $w_0 \to 0$ we have $\Delta' \to |w|/|1 + w|$, $\widetilde{\Delta} \to |w|/|1 + w|^{1/2}$, $\Psi \to |1 + w|$ and $\lambda = \Delta'/\widetilde{\Delta} \to 1/|1 + w|^{1/2}$. So the comparison of the bounds is the same as for $G = K_2$, and Theorem 1.2 beats Theorem 1.3 by a factor tending to $K_0^*/4 \approx 1.223222$ as $|w| \to \infty$.

For instance, let G be the n-cycle C_n with $n \geq 3$, taking $w_e = w$ for exactly one edge and $w_e = w_0$ for all other edges. Then $Z_G(q,w) = (q+w)(q+w_0)^{n-1} + ww_0^{n-1}(q-1)$. As $|w| \to \infty$ at fixed n and w_0, we have $Q_{\max}(G,\boldsymbol{w}) = |w| + o(|w|)$. On the other hand, if $|1+w|, |1+w_0| \geq 1$ we have $\Delta'(G,\boldsymbol{w}) = |w|/|1+w| + |w_0|/|1+w_0|$, $\widetilde{\Delta}(G,\boldsymbol{w}) = |w|/|1+w|^{1/2} + |w_0|/|1+w_0|^{1/2}$ and $\Psi(G,\boldsymbol{w}) = |1+w_0|\,|1+w|$. Therefore, as $|w| \to \infty$ the bounds of Theorems 1.2 and 1.3 are $4(|w_0| + |1+w_0|)|w| + O(1)$ and $K_0^* |1+w_0|^{1/2}|w| + O(|w|^{1/2})$, respectively, where $K_0^* \approx 4.892888$. Both of these bounds have the correct magnitude as $|w| \to \infty$ at fixed n and w_0. The bound given by Theorem 1.2 is better than that given by Theorem 1.3 when $|w_0|$ is small, and worse when $|w_0|$ is large. $\square$

Example 7.3 Let $G = K_2^{(k)}$ (a pair of vertices connected by k parallel edges) with $w_e = w$ for all e. Then $Z_G(q,w) = q[q + (1+w)^k - 1]$, so $Q_{\max}(G,\boldsymbol{w}) = |(1+w)^k - 1|$. Now, if $|1+w| \geq 1$ we have $\Delta'(G,\boldsymbol{w}) = k|w|/|1+w|$ and $\Psi(G,\boldsymbol{w}) = |1+w|^k$. Therefore, as $|w| \to \infty$ at fixed k, the bound of Theorem 1.2 is a factor $4k$ from being sharp.

On the other hand, we may first apply parallel reduction to yield a simple graph $\widehat{G} = K_2$ with weight $\widehat{w} = (1+w)^k - 1$ on its single edge, and then apply Theorem 1.2 or 1.3 to $(\widehat{G}, \widehat{\boldsymbol{w}})$. The resulting bound is then (as $|w| \to \infty$) a factor 4 or ≈ 4.892888 from being sharp (see Example 7.1). $\square$

Example 7.4 Let G be the n-cycle C_n (which is simple for $n \geq 3$), with $w_e = w$ for all e. Then $Z_G(q,w) = (q+w)^n + (q-1)w^n$. As $|w| \to \infty$ at fixed n, we have $Q_{\max}(G,\boldsymbol{w}) = |w|^{n/(n-1)} + O(|w|)$. On the other hand, if $|1+w| \geq 1$ we have $\Delta'(G,\boldsymbol{w}) = 2|w|/|1+w|$, $\widetilde{\Delta}(G,\boldsymbol{w}) = 2|w|/|1+w|^{1/2}$ and $\Psi(G,\boldsymbol{w}) = |1+w|^2$. Therefore, as $|w| \to \infty$ the bounds of Theorems 1.2 and 1.3 are $8|w|^2 + O(|w|)$ and $2K_0^*|w|^{3/2} + O(|w|)$, respectively (here $2K_0^* \approx 9.785776$). Both of these bounds have the wrong order of magnitude as $|w| \to \infty$ at fixed $n \geq 4$, but the bound given by Theorem 1.3 is a significant improvement over that given by Theorem 1.2. $\square$

Example 7.5 Let G be the complete graph K_n. Take $w_e = w > 0$ for all e, with w fixed independent of n (unlike the usual [8] scaling $w = \lambda/n$). Then Janson [18] has very recently proven that

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_{K_n}(e^{\alpha n}, w) \;=\; \max[\tfrac{1}{2}\log(1+w), \alpha] \qquad \text{for } \alpha \geq 0\,. \tag{7.7}$$

[This is because the sum (1.1) is dominated by two contributions: the terms with (V,A) connected, which together contribute $e^{\alpha n}(1+w)^{\binom{n}{2}}[1+o(1)]$, and the term $A = \varnothing$, which contributes $e^{\alpha n^2}$.] It then follows from the Yang–Lee [35] theory of phase transitions (see e.g. [28, Theorem 3.1]) that $Z_{K_n}(e^{\alpha n}, w)$ must have complex roots α_n that converge to $\alpha_\star = \frac{1}{2}\log(1+w)$ as $n \to \infty$. Hence $Q_{\max}(K_n, \boldsymbol{w}) \geq (1+w)^{n/2 + o(n)}$

(and this is presumably the actual order of magnitude). On the other hand, we have $\widetilde{\Delta}(K_n, \boldsymbol{w}) = (n-1)w/(1+w)^{1/2}$ and $\Psi(G, \boldsymbol{w}) = (1+w)^{n-1}$, so that the upper bound given by Theorem 1.3 is nearly sharp [it exceeds the truth by at most a factor $O(n)$ even though both the truth and the bound are growing exponentially in n]. By contrast, the bound of Theorem 1.2 is much worse, because of its growth as $\Psi(G, \boldsymbol{w})$ rather than $\Psi(G, \boldsymbol{w})^{1/2}$. $\square$

Example 7.6 Let G be a large finite piece of the simple hypercubic lattice $\mathbb{Z}^d$ (for some fixed $d \geq 2$) with nearest-neighbor edges, and take $w_e = w > 0$ for all e (here $w > 0$ corresponds to the ferromagnetic case). For real $q > 0$ sufficiently large, it is known [23, 22, 19, 21, 10] that the first-order phase-transition point w_t lies at

$$w_t(q) \;=\; q^{1/d} \,+\, O(1)\,. \tag{7.8}$$

It then follows from the Yang–Lee [35] theory of phase transitions that there will be complex zeros of the partition function arbitrarily close (as G grows) to the phase-transition point $(q, w_t(q))$; so as $w \uparrow \infty$ (for fixed $d \geq 2$) we will have asymptotically $Q_{\max}(G, \boldsymbol{w}) \geq w^d[1+O(1/w)]$ (and this is presumably the actual order of magnitude). Since $\widetilde{\Delta}(G, \boldsymbol{w}) = 2dw/(1+w)^{1/2}$ and $\Psi(G, \boldsymbol{w}) = (1+w)^{2d}$, the upper bound given by Theorem 1.3 is off by at most a factor of order $w^{1/2}$ (i.e. it gives a growth $\sim w^{d+1/2}$ instead of w^d). It is perhaps worth observing that the bound would be off by only a bounded factor if Theorem 1.3 could be improved to use $\Delta'(G, \boldsymbol{w})$ rather than $\widetilde{\Delta}(G, \boldsymbol{w})$. By contrast, the bound of Theorem 1.2 is again much worse, because of its growth as $\Psi(G, \boldsymbol{w})$ rather than $\Psi(G, \boldsymbol{w})^{1/2}$. $\square$

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